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PRIMARY SUMMAND FUNCTIONS ON THREE-DIMENSIONAL COMPACT SOLVMANIFOLDS

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ABSTRACT. Leonard Richardson has shown that for a certain class of three-dimensional compact solvmanifolds, projections onto π-primary summands of $L^2(M)$ do not preserve the continuity of functions on M. It is shown here that if the π-primary projection of a continuous function is L^∞ then it is actually continuous. From this it follows that there are continuous functions on M whose π-primary projections are essentially unbounded.

INTRODUCTION

Let G be a solvable, connected, and simply connected Lie group, with Lie algebra $\mathbf{g}$ and cocompact discrete subgroup Γ. By a representation π of G we shall mean a strongly continuous, unitary representation of G in some separable Hilbert space H_π; π will be called irreducible if the space H_π contains no proper closed nontrivial subspace invariant under π.

Let M be the space of right Γ cosets in G and give M the quotient topology. There is a unique probability measure on M that is invariant under the natural action of G by right translation. Forming $L^2(M)$ with respect to this measure, we have that the action of G on M induces a unitary action of G on $L^2(M)$: $R(g)f(m) = f(mg)$ for $f \in L^2(M)$, $m \in M$, and $g \in G$. R is called the quasi-regular representation of G on $L^2(M)$.

It is well known that $L^2(M)$ decomposes into the direct sum $\bigoplus H_\pi$, where the spaces H_π are mutually orthogonal $R(G)$-invariant subspaces, and that R on the space H_π is a finite multiple of the irreducible representation π. [GGP, §I.2]. We let $(\Gamma\backslash G)^\wedge$ denote the set of irreducible representations appearing in the quasi-regular representation R of G on $L^2(M)$. Then the orthogonal projection P_π of $L^2(M)$ onto H_π is L^2-continuous and preserves $C^\infty(M)$ [AB, Theorem 5].

Now let N be a nilpotent Lie group, connected and simply connected, with Lie algebra $\mathbf{n}$ and cocompact discrete subgroup Γ.

If the coadjoint orbits of the action of N on the dual $\mathbf{n}^*$ are linear varieties, then $\Gamma\backslash N$ possesses the property that the orthogonal projections P_π of $L^2(\Gamma\backslash N)$ onto H_π preserve continuity [Ri1, B]. These flat-orbit nilmanifolds

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share this property with compact quotients of the three-dimensional solvable group S_R by discrete subgroups. Here S_R denotes the semidirect product $\mathbf{R} \propto \mathbf{R}^2$, where $\mathbf{R}$ acts on $\mathbf{R}^2$ via a one-parameter subgroup of rotations [Ri1].

This work was motivated by the fact that orthogonal projections onto π-primary summands preserves continuity of functions in L^2 of both compact quotients of flat-orbit nilmanifolds and compact quotients of the group S_R. In [Ri2] Richardson proved a Fejer theorem for flat-orbit nilmanifolds. It can be shown that a similar Fejer theorem holds for compact quotients of S_R by discrete subgroups. Let S_H be the semidirect product $\mathbf{R} \propto \mathbf{R}^2$, where $\mathbf{R}$ acts upon $\mathbf{R}^2$ via the one-parameter subgroup $t \mapsto [\begin{smallmatrix} \lambda^t & \\ & \lambda^{-t} \end{smallmatrix}]$ in $\mathrm{SL}_2(\mathbf{R})$, where $\lambda + \lambda^{-1} = k + 1$ for some integer $k \geq 2$. Let Γ be a cocompact discrete subgroup of S_H. It is known that orthogonal projections P_π of $L^2(\Gamma\backslash S_H)$ onto H_π do not preserve continuity [Ri1]. It can be shown that no standard Fejer theorem exists for solvmanifolds of this type. However, the series determining the L^2 equivalence class of a projected function bears a resemblance to a standard lacunary Fourier series on $\mathbf{R}\backslash\mathbf{Z}$ (see [Z]). An adaptation of Sidon's theorem on convergence of lacunary Fourier series [Z, Theorem VI.6.1] is used to demonstrate that if the orthogonal projection $P_\pi f$ of a continuous function f on $\Gamma\backslash S_H$ is an L^∞ function, then $P_\pi f$ is actually continuous. Together with Theorem 3.13 in [Ri1], this implies that for each H_π there is continuous $f \in L^2(\Gamma\backslash S_H)$ such that $P_\pi f$ is discontinuous and essentially unbounded.

Preliminaries

Let G be a connected, simply connected Lie group with Lie algebra $\mathbf{g}$, and let $\mathbf{g}^*$ be the vector space of linear functionals on $\mathbf{g}$. We define a sequence of ideals of the Lie algebra $\mathbf{g}$ by $\mathbf{g}^{(0)} = \mathbf{g}$, $\mathbf{g}^{(k)} = [\mathbf{g}^{(k-1)}, \mathbf{g}^{(k-1)}]$; this is called the derived series of $\mathbf{g}$, and $\mathbf{g}$ is said to be solvable if $\mathbf{g}^{(n)} = 0$ for some $n \in \mathbf{N}$. We define another sequence of ideals of the Lie algebra $\mathbf{g}$ by $\mathbf{g}_{(0)} = \mathbf{g}$, $\mathbf{g}_{(k)} = [\mathbf{g}_{(k-1)}, \mathbf{g}]$; this is called the lower central series of $\mathbf{g}$, and $\mathbf{g}$ is said to be nilpotent if $\mathbf{g}_{(n)} = 0$ for some $n \in \mathbf{N}$ (see [H, §3]. Throughout this paper, the term "nilmanifold" ("solvmanifold") will refer to compact spaces $\Gamma\backslash G$, where G is nilpotent (solvable) and Γ is discrete and cocompact.

The coadjoint representation of G is of central importance in the representation theory of nilpotent and solvable Lie groups. The set of equivalence classes of irreducible representations of a nilpotent Lie group G is naturally parametrized by the orbit space $\mathbf{g}^*/\mathrm{Ad}^*G$; this is also true for the (completely) solvable Lie groups examined in this work. This parametrization, due to A. A. Kirillov, is freely drawn upon in this work; for details, see [CG, Chapter II].

There are two three-dimensional, solvable, nonnilpotent Lie groups with cocompact discrete subgroups, the groups S_H and S_R. Their Lie algebras are three-dimensional vector spaces spanned by the vectors T, X, and Y, where $\exp sT = (s, 0, 0)$, $\exp sX = (0, s, 0)$, and $\exp sY = (0, 0, s)$.

There are five distinct compact quotients of S_R and infinitely many distinct compact quotients of S_H.

We have the following compact quotients of S_H, with convenient coordinatizations.

Suppose $k \in \mathbf{Z}$, $k \geq 2$. Define $S_{H,k} = \mathbf{R} \propto \mathbf{R}^2$, where $\mathbf{R}$ acts on $\mathbf{R}^2$ via

the one-parameter subgroup $\sigma_k(t)$ in $\mathrm{SL}_2(\mathbf{R})$ with $\sigma_k(1) = \left[\begin{smallmatrix} 1 & 1 \\ k-1 & k \end{smallmatrix}\right]$. (We will henceforth refer to the normal subgroup $\mathbf{R}^2$ as N.) Then $S_{H,k} \cong S_H$ for each k. Let $\Gamma_{H,k} = \{(p, m, n) \in S_{H,k};\ p, m, n \in \mathbf{Z}\}$; then each $\Gamma_{H,k}\backslash S_{H,k} = M_{H,k}$ is a distinct compact quotient of S_H.

The coordinatizations of the solvmanifolds $S_{H,k}$ just described will be referred to as the integral coordinatizations of $S_{H,k}$. Let $A \in \mathrm{GL}_2(\mathbf{R})$ be such that

$$A\sigma_k(t)A^{-1} = \begin{bmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{bmatrix} \quad \text{where } \lambda + \lambda^{-1} = k + 1;$$

if we recoordinatize N so that the action of $\mathbf{R}$ on N is given by this one-parameter subgroup, then $\Gamma_{H,k} \cap N = A(\mathbf{Z}^2)$; the nondegenerate coadjoint orbits in this case are hyperbolic cylinders of the form $xy = \lambda$, $\lambda \in \mathbf{R}$. The 2-torus $N \cap \Gamma_{H,k}\backslash N$ in this coordinatization will be a nonstandard torus, for all $k \geq 2$. This coordinatization of $S_{H,k}$ will be referred to as the hyperbolic coordinatization. We henceforth fix $S_{H,k}$ and refer to it as S and to $\Gamma_{H,k}$ as Γ.

Consider the quasi-regular representation of S on $L^2(M)$. Since M is a compact solvmanifold, $L^2(M)$ decomposes canonically into the discrete direct sum of subspaces H_π, which are invariant with respect to the action of S, and are such that when the action of S is restricted to the subspace H_π, S acts on H_π as some finite multiple of an irreducible representation π of S. There exists no canonical decomposition of H_π into irreducible subspaces unless H_π is itself irreducible.

$(\Gamma\backslash S)^\wedge$ will denote the set of unitary irreducible representations of S appearing in the quasi-regular representation R of S in $L^2(\Gamma\backslash S)^\wedge$.

$(\Gamma\backslash S)^\wedge_\infty$ will denote the set of those representations $\pi \in (\Gamma\backslash S)^\wedge$ that are infinite dimensional.

In the integral coordinatization of S, the coadjoint orbits satisfy

$$(k-1)x^2 + (k-1)xy - y^2 = K, \tag{1}$$

so that the orbits are saturated in the T^*-direction. We will call λ an integral functional if $\lambda|_{\mathbf{n}} = \alpha X^* + \beta Y^*$, $\alpha, \beta \in \mathbf{Z}$, and denote by $\mathscr{O}_\pi$ the orbit of λ in s^*.

Fix some nonzero integral functional $\lambda \in \mathscr{O}_\pi$. We define the character χ_λ on the (abelian) nilradical $\mathbf{n}$ as follows: if $\lambda|_{\mathbf{n}} = \alpha X^* + \beta Y^*$ then

$$\chi_\lambda(0, r, s) = e^{2\pi i(\alpha r + \beta s)}. \tag{2}$$

In the hyperbolic coordinatization of S, integral functionals are elements of a nonstandard lattice obtained by transforming $\mathbf{Z}^2$. Let $\{\lambda_i\}_{i=1}^{\mathrm{mul}\,\pi}$ be a set of representatives of distinct Γ-orbits in the integral functionals of $\mathscr{O}_\pi$, λ_i an integral functional for each i. Then for f a continuous function on the solvmanifold, we have

$$P_\pi(f)(\Gamma(t, x, y)) = \sum_{i=1}^{\mathrm{mul}\,\pi} \sum_{n \in \mathbf{Z}} \tilde{f}_i(n+t)\chi_{\gamma_k^*(n)\lambda_i}(0, x, y) \tag{3}$$

where for fixed t, $\tilde{f}_i(n+t)$ is the Fourier coefficient of f at $\gamma_k^*(n)\lambda_i$ and where $\gamma_k(n) = A\sigma_k(n)A^{-1}$.

1. A Sidon theorem for primary summand functions in $H_\pi \subset L^2(M)$

Suppose $S = \mathbf{R} \propto \mathbf{N}$ is coordinatized so that $\mathbf{R}$ acts on N via the one-parameter subgroup $\sigma_H(t) = \begin{bmatrix} \lambda^t & \\ & \lambda^{-t} \end{bmatrix}$. Then the coadjoint orbits will be "hyperbolic cylinders," saturated in the t-direction, given by the equations $xy = k$, $k \in \mathbf{R}$. Let $\pi \in (\Gamma\backslash S)^\wedge$ be an infinite-dimensional representation. Let $P_\pi\colon L^2(\Gamma\backslash S) \to L^2(\Gamma\backslash S)$ be the orthogonal projection onto the π-primary summand of L^2; P_π does not preserve continuity of functions [Ri1]. Let (α, β) be a fixed lattice point in the coadjoint orbit $\mathscr{O}_\pi$, lying in the plane $\mathbf{R}X^* + \mathbf{R}Y^*$, noting that with the chosen coordinatization of S, the torus $N \cap \Gamma\backslash N$ will be a nonstandard torus, and so $(\alpha, \beta) \in \mathscr{O}_\pi$ satisfying $\chi_{(\alpha,\beta)}(N \cap \Gamma) = 1$ will not have integer coordinates. We will call the set of (α', β') satisfying $\chi_{(\alpha',\beta')}(N \cap \Gamma) = 1$ the lattice $\mathscr{L}^*$. $\mathscr{L}^* \cap \mathscr{O}_\pi$ consists of finitely many Γ orbits of integral points.

Theorem 1.1. *$\mathscr{L}^* \cap \mathscr{O}_\pi$ is a Sidon set for each $\pi \in (\Gamma\backslash S)^\wedge_\infty$.*

Proof. See [Ru, p. 127].

Corollary 1.2. *Suppose $f \in P_\pi(L^2(M))$ and $f \in L^\infty(M)$. Then for almost all fixed $t = t_0$, we have*

$$f(\Gamma(t_0, x, y)) = \sum_{(\alpha,\beta)\in\mathscr{O}_\pi\cap\mathscr{L}^*} f(t_0, \cdot, \cdot)^\wedge(\alpha, \beta)\chi_{(\alpha,\beta)}(x, y)$$

absolutely and uniformly convergent to f.

Proof. Follows from the definition of a Sidon set.

Corollary 1.3. *Suppose $\pi \in (\Gamma\backslash S)^\wedge_\infty$ and that $f \in L^2(\Gamma\backslash S)$ is continuous on $\Gamma\backslash S$. If $P_\pi f$ is L^∞ then $P_\pi f$ is continuous.*

Proof. Let $R = \{(t, 0, 0) \in S\colon t \in \mathbf{R}\}$. Then R is a subgroup of S.

Suppose $f \in L^2(\Gamma\backslash S)$ is continuous. Then for $(\alpha, \beta) \in \mathscr{L}^*$, we have

$$f(t, \cdot, \cdot)^\wedge(\alpha, \beta) = \int_{T^2} f(t, x, y)\chi_{-(\alpha,\beta)}(x, y)\,dx\,dy$$

continuous on R.

If $P_\pi f$ is in L^∞ then we have

$$\sum_{(\alpha,\beta)\in\mathscr{O}_\pi\cap\mathscr{L}^*} |f(t, \cdot, \cdot)^\wedge(\alpha, \beta)| < K\|f\|_\infty \tag{4}$$

where K is a constant not depending on f.

Since the inequality is independent of t, we have that

$$P_\pi f = \sum_{(\alpha,\beta)\in\mathscr{O}_\pi\cap\mathscr{L}^*} f(t, \cdot, \cdot)^\wedge(\alpha, \beta)\chi_{-(\alpha,\beta)}(x, y) \tag{5}$$

is the uniformly convergent sum of functions continuous on S. Since $P_\pi f$ also possesses left Γ-invariance, $P_\pi f$ is continuous on $\Gamma\backslash S$.

Corollary 1.4. *There exists f, continuous on M, such that $P_\pi f$ is essentially unbounded.*

Proof. Example 5.3 in [Ri1] shows that for each orthogonal projection P_π, there must exist an $f \in C(M)$ such that $P_\pi f$ is not continuous. By Corollary 1.3, however, $P_\pi f$ must then be essentially unbounded.

Corollary 1.5. *If $f \in P_\pi(L^2(M))$ is L^∞, then for a.e. fixed t_0 we have that*

$$f(t_0, \cdot, \cdot): N \cap \Gamma\backslash N \to \mathbf{C}$$

is a continuous function on $N \cap \Gamma\backslash N \cong T^2$.

Proof. By Corollary 1.2, we have that for $f \in P_\pi(L^2(\Gamma\backslash S))$ essentially bounded, inequality (4) holds, for a.a. t_0.

Thus for a.a. fixed t_0,

$$f(\Gamma(t_0, x, y)) = \sum f(t_0, \cdot, \cdot)^\wedge(\alpha, \beta)\chi_{(\alpha,\beta)}(x, y)$$

is a uniformly convergent sum of functions that are continuous on $N \cap \Gamma\backslash N$ and so is itself continuous on $N \cap \Gamma\backslash N$.

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