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ONE RADIUS THEOREM FOR THE EIGENFUNCTIONS OF THE INVARIANT LAPLACIAN

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ABSTRACT. Let B be the open unit ball in $\mathbf{C}^n$ with its boundary S. Suppose that $\alpha \geq \frac{1}{2}$ and $u(z) = (1-|z|^2)^{n(1-\alpha)}F(z)$ for some $F(z) \in C(\overline{B})$. If for every $z \in B$ there corresponds an $r(z)$: $0 < r(z) < 1$ and an automorphism ψ_z with $\psi_z(0) = z$ such that

$$u(z) = \frac{1}{g_\alpha(r(z))} \int_S u \circ \psi_z(r(z)\zeta)\, d\sigma(\zeta),$$

then $\widetilde{\Delta} u(z) = -4n^2\alpha(1-\alpha)u(z)$, $z \in B$. Here $\widetilde{\Delta}$ is the invariant Laplacian and $g_\alpha(r)$ is the hypergeometric function $F(n-n\alpha, n-n\alpha, n; r^2)$.

1. INTRODUCTION

Let ψ_z be an automorphism, that is, a one-to-one holomorphic map onto itself, of the open unit ball B of $\mathbf{C}^n$ satisfying $\psi_z(0) = z$. Let ϕ_z be one of such an automorphism defined by

$$\phi_z(w) = \frac{z - P_z w - s_z(w - P_z w)}{1 - \langle w, z\rangle}, \qquad w \in B,$$

where P_z is the orthogonal projection of $\mathbf{C}^n$ onto the subspace generated by z, $\langle w, z\rangle = \sum_{j=1}^n w_j \overline{z}_j$, $s_z = \sqrt{(1-|z|^2)}$, and $|z|^2 = \langle z, z\rangle$. For $u \in C^2(B)$, its *invariant Laplacian* $\widetilde{\Delta}$ is well defined by

$$(\widetilde{\Delta} u)(z) = \Delta(u \circ \psi_z)(0), \qquad z \in B,$$

where Δ is the ordinary Laplacian. Our starting point is the following *mean value theorem* [4, Theorem 4.2.4].

Theorem A. *Given a complex number λ, any function $u(z) \in C^2(B)$ satisfying*

$$\widetilde{\Delta} u(z) = \lambda u(z), \qquad z \in B, \tag{1}$$

has mean value property

$$g_\alpha(r)u(z) = \int_S u \circ \psi_z(r\zeta)\, d\sigma(\zeta), \qquad z \in B,\ 0 \leq r < 1. \tag{2}$$

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Here and throughout λ and α are related to be

$$\lambda = \lambda_\alpha = -4n^2\alpha(1-\alpha),$$

$d\sigma$ is the normalized rotation invariant positive Borel measure on the boundary S of B, and

$$g_\alpha(r) = \int_S \left(\frac{1-r^2}{|1-r\zeta_1|^2}\right)^{n\alpha} d\sigma(\zeta).$$

The space of all $u(z)$ satisfying (1) is denoted by X_λ. This space is invariant under automorphism. See [4, Chapter 4] for the terminology and related properties.

When $\lambda = 0$, Theorem A has a strong converse known as the *one radius theorem* [4, Theorem 4.3.4]. See [2] also for the same vein. Motivated by these, our goal here is in the description of the complex numbers α and the smoothness of functions $u(z)$ that makes the following property valid.

One-radius property for α and u. *If for every $z \in B$ there corresponds an $r(z)$: $0 < r(z) < 1$ such that*

$$g_\alpha(r(z))u(z) = \int_S u \circ \phi_z(r(z)\zeta)\, d\sigma(\zeta),$$

then $u(z) \in X_\lambda$.

We denote this property by $1\mathrm{RP}(\alpha; u)$ in short. Rudin's *one radius theorem* says that $1\mathrm{RP}(0; u) = 1\mathrm{RP}(1; u)$ is true if $u(z) \in C(\overline{B})$. Izuchi [1] proved that $1\mathrm{RP}(1; u)$ fails for a bounded real analytic $u(z)$. In this paper, we first confine ourselves to $\mathrm{Re}\,\alpha > \frac{1}{2}$. Our results say that under our growth condition on u, $1\mathrm{RP}(\alpha; u)$ is true if and only if α is real (Theorem 3), and the growth condition on u cannot be improvable (Theorem2). Our last section deals with the case $\alpha = \frac{1}{2}$. All unexplained notations and properties will be referred to [4].

2. One radius theorem for X_λ

Lemma 1. *Let $\alpha \in \mathbf{C}$. Then*

$$g_\alpha(r) = g_{1-\alpha}(r) = (1-r^2)^{n(1-\alpha)}\mathscr{F}(\alpha : r), \tag{3}$$

where

$$\mathscr{F}(\alpha : r) = \int_S \frac{(1-r^2)^{2n\alpha-n}}{|1-r\zeta_1|^{2n\alpha}}\, d\sigma(\zeta).$$

If α is real then $\mathscr{F}(\alpha : r)$ is an increasing function of r. If $\alpha > \frac{1}{2}$ then $\mathscr{F}(\alpha : r)$ tends to a finite limit $\mathscr{F}(\alpha) = \lim_{r\to 1}\mathscr{F}(\alpha : r)$. Also

$$\mathscr{G}\left(\frac{1}{2} : r\right) = \frac{1}{r^2}\log\frac{1}{1-r^2}\mathscr{F}\left(\frac{1}{2} : r\right)^{-1}$$

is a continuous function of r that tends to a finite limit $\mathscr{G}(\frac{1}{2}) = \lim_{r\to 1}\mathscr{G}(\frac{1}{2} : r)$.

Theorem 1. *Suppose $\alpha > \frac{1}{2}$ and $u(z) = (1-|z|^2)^{n(1-\alpha)}F(z)$ for some $F(z) \in C(\overline{B})$. If for every $z \in B$ there corresponds an $r(z)$: $0 < r(z) < 1$ and an*

automorphism ψ_z *such that*

$$g_\alpha(r(z))u(z) = \int_S u \circ \psi_z(r(z)\zeta)\, d\sigma(\zeta), \tag{4}$$

then

$$u(z) = \frac{1}{\mathscr{F}(\alpha)} \int_S P^\alpha(z, \zeta) F(\zeta)\, d\sigma(\zeta), \qquad z \in B,$$

where $P(z, w)$ *denotes the invariant Poisson kernel. In particular,* $1\mathrm{RP}(\alpha; u)$ *is true.*

Proof of Lemma 1. By [4, Remark, p. 44],

$$\begin{aligned}\mathscr{F}(\alpha : r) &= \int_S \frac{(1-r^2)^{2n\alpha-n}}{|1-\langle r\eta, \phi_{r\eta}(\zeta)\rangle|^{2n\alpha}} P(\phi_{r\eta}^{-1}(0), \zeta)\, d\sigma(\zeta) \\ &= \int_S \frac{(1-r^2)^{2n\alpha-n}}{|1-\langle \phi_{r\eta}(0), \phi_{r\eta}(\zeta)\rangle|^{2n\alpha}} \frac{(1-r^2)^n}{|1-\langle r\eta, \zeta\rangle|^{2n}}\, d\sigma(\zeta),\end{aligned}$$

which turns out to be

$$\int_S \frac{d\sigma(\zeta)}{|1-\langle r\eta, \zeta\rangle|^{2n(1-\alpha)}}$$

once we use the identity [4, Theorem 2.2.5]

$$1 - \langle \psi_a(z), \psi_a(w)\rangle = \frac{(1-|a|^2)(1-\langle z, w\rangle)}{(1-\langle z, a\rangle)(1-\langle a, w\rangle)}, \tag{5}$$

$a \in B$, $z, w \in \overline{B}$. Hence $(1-r^2)^{n(1-\alpha)}\mathscr{F}(\alpha : r) = g_{1-\alpha}(r)$. That $g_\alpha(r) = g_{1-\alpha}(r)$ is noted in [4, Corollary, p. 51]. The second part follows from the subharmonicity of the integrand and [4, Proposition 1.4.10].

Proof of Theorem 1. Let

$$v(z) = \frac{1}{\mathscr{F}(\alpha)} \int_S P^\alpha(z, \zeta) F(\zeta)\, d\sigma(\zeta), \qquad z \in B.$$

Then $v(z) \in X_\lambda$ because $P^\alpha(z, \zeta) \in X_\lambda$ [4, Theorem 4.2.2], so that it suffices to show that $h(z) \equiv u(z) - v(z) \equiv 0$. Consider first the function defined by

$$G(z) = \begin{cases} (1-|z|^2)^{n(\alpha-1)} v(z), & z \in B, \\ F(z), & z \in S. \end{cases}$$

For $\zeta \in S$,

$$\begin{aligned}|G(r\zeta) - G(\zeta)| &\le \left|G(r\zeta) - \frac{\mathscr{F}(\alpha : r)}{\mathscr{F}(\alpha)} F(\zeta)\right| + |F(\zeta)| \left|\frac{\mathscr{F}(\alpha : r)}{\mathscr{F}(\alpha)} - 1\right| \\ &= |(I)| + |(II)|.\end{aligned}$$

Since $F(z) \in C(S)$, by Lemma 1, $(II) \to 0$ as $r \to 1$ uniformly on S. Also since

$$(I) = \frac{1}{\mathscr{F}(\alpha)} \int_S \frac{(1-r^2)^{2n\alpha-n}}{|1-\langle r\zeta, \eta\rangle|^{2n\alpha}} [F(\eta) - F(\zeta)]\, d\sigma(\eta),$$

by dividing S into two parts—a small neighborhood Q of ζ and $S - Q$—it is routine to see that $(I) \to 0$ as $r \to 1$ uniformly on S. Therefore $G(z) \in C(\overline{B})$.

Now the function $H(z)$ defined by

$$H(z) = \begin{cases} (1-|z|^2)^{n(\alpha-1)}h(z), & z \in B, \\ 0, & z \in S, \end{cases}$$

is of $C(\overline{B})$. We will show $H \equiv 0$ in B. Suppose $|H(z)| > 0$ for some $z \in B$. Then the set E on which $|H(z)|$ takes the maximum is compact. We can take $z_0 \in E$ such that $\operatorname{dist}(z_0, S) = \operatorname{dist}(E, S)$. Since both $u(z)$ and $v(z)$ satisfy (4), we have (4) with $h(z)$ in place of $u(z)$, so that

$$H(z)g_\alpha(r(z)) = \int_S (1-|z|^2)^{n(\alpha-1)}\, h \circ \psi_z(r(z)\zeta)\, d\sigma(\zeta), \qquad z \in B. \tag{6}$$

But the right side of (6) is

$$\int_S P^{1-\alpha}(r(z)\zeta, z)\, H \circ \psi_z(r(z)\zeta)\, d\sigma(\zeta) \tag{7}$$

once we use the identity (5). If we apply (6) and (7) to z_0 with its corresponding $r(z_0) = r_0$, we obtain, by (3),

$$|H(z_0)| = \frac{1}{\mathscr{F}(\alpha : r_0)} \left| \int_S \frac{H \circ \psi_{z_0}(r_0\zeta)}{|1 - \langle r_0 z_0, \zeta\rangle|^{2n(1-\alpha)}}\, d\sigma(\zeta) \right|,$$

which is, by (3) once more, strictly less than

$$\frac{\mathscr{F}(\alpha : r_0|z_0|)}{\mathscr{F}(\alpha : r_0)} |H(z_0)| \le |H(z_0)|. \tag{8}$$

This contradiction proves our assertion.

3. On smoothness

Because of the symmetry, $g_\alpha = g_{1-\alpha}$ and $X_{\lambda_\alpha} = X_{\lambda_{1-\alpha}}$, we can have the proper substitute for the case $\alpha < \frac{1}{2}$, that is, Theorem 1 with all α replaced by $1-\alpha$. On the other hand, if $\alpha < \frac{1}{2}$ and $(1-|z|^2)^{n(\alpha-1)}u(z)$ can be of $C(\overline{B})$, then we can easily check from Theorem 1 that $1\mathrm{RP}(\alpha; u) = 1\mathrm{RP}(1-\alpha; u)$ is true. But if $\alpha \le \frac{1}{2}$, there is no $u \in X_\lambda$, $u \not\equiv 0$ with $(1-|z|^2)^{n(\alpha-1)}u(z)$ bounded. This can be checked by comparing the order of $g_\alpha(r)$ as $r \to 1$ after integrating $(1-|\psi_z(r\zeta)|^2)^{n(\alpha-1)}u \circ \psi_z(r\zeta)$ with respect to $d\sigma(\zeta)$ and using (2) in Theorem A and (5). This fact and Theorem 1 gives the following solution to the boundary value problem.

Corollary. *Let α be real and let $f \in C(S)$, $f \not\equiv 0$, be given. Then there is a solution $u(z)$ such that*

(i) $\widetilde{\Delta} u(z) = \lambda u(z)$ *in* B,

(ii) *with boundary value* $f(\zeta)$, $(1-|z|^2)^{n(\alpha-1)}u(z)$ *is of* $C(\overline{B})$

if and only if $\alpha > \frac{1}{2}$. *In fact,*

$$u(z) = \frac{1}{\mathscr{F}(\alpha)} \int_S P^\alpha(z, \zeta) f(\zeta)\, d\sigma(\zeta)$$

is the unique solution when $\alpha > \frac{1}{2}$.

Our proof of Theorem 1 covers the case $\alpha = 1$. But in (8) equality occurs only when $\alpha = 1$ because $\mathscr{F}(\alpha : r)$ is strictly increasing if α is real and $\alpha \ne 1$.

So it seems to be that the compactness of E when $\alpha \neq 1$ is not so much crucial compared to the case $\alpha = 1$. Nonetheless we can explain by use of an example of Izuchi that our growth condition cannot be weakened.

Theorem 2. *For $\alpha > \frac{1}{2}$, there is a function $u(z)$ such that*
(i) $u(z) \notin X_\lambda$,
(ii) $(1-|z|^2)^{n(\alpha-1)}u(z)$ *is bounded real analytic in* B,
(iii) *for every* $z \in B$, *there is a radius* $r(z) : 0 < r(z) < 1$ *by which* $u(z)$ *satisfies* (4).

Proof. Izuchi [1] constructed a bounded positive radial real analytic function $F(z)$ on B such that

$$\sup_{\zeta \in S} F \circ \phi_z(\delta_j \zeta) < F(z) < \inf_{\zeta \in S} F \circ \phi_z(\sigma_j \zeta) \tag{9}$$

for some δ_j, σ_j depend on z (see [1, p. 830]). If we fix α and set $u(z) = g_\alpha(z)F(z)$, then by (9) and (2),

$$\int_S u \circ \phi_z(\delta_j \zeta)\, d\sigma(\zeta) < F(z) \int_S g_\alpha \circ \phi_z(\delta_j \zeta)\, d\sigma(\zeta) = u(z) g_\alpha(\delta_j) \tag{10}$$

and

$$u(z) g_\alpha(\sigma_j) = F(z) \int_S g_\alpha \circ \phi_z(\sigma_j \zeta)\, d\sigma(\zeta) < \int_S u \circ \phi_z(\sigma_j \zeta)\, d\sigma(\zeta),$$

so that by the continuity on r of

$$\frac{1}{g_\alpha(r)} \int_S u \circ \phi_z(r\zeta)\, d\sigma(\zeta),$$

(iii) is valid. This $u(z)$ satisfy (i), because of the strict inequality in (10), and (ii) also, because $F(z)$ and $(1-|z|^2)^{n(\alpha-1)}g_\alpha(z)$ are bounded real analytic.

4. One-radius property fails for nonreal α

Lemma 2. (1) *If $\frac{1}{2} \le a_1 < a_2$ then $g_{a_1}(r) < g_{a_2}(r)$, $0 < r < 1$.*
(2) *For a given nonreal β with $\operatorname{Re}\beta > \frac{1}{2}$, there exist r and a such that* (i) $0 < r < 1$, (ii) $\frac{1}{2} < a < \operatorname{Re}\beta$, *and* (iii) $g_a(r) = g_\beta(r)$.

Theorem 3. *Let $\operatorname{Re}\alpha > \frac{1}{2}$. Then the following are equivalent:*
(1) $1\mathrm{RP}(\alpha; u)$ *is true for all* $u(z)$ *with* $(1-|z|^2)^{n(\alpha-1)}u(z) \in C(\overline{B})$.
(2) α *is real.*

Proof of Lemma 2. (1) Suppose $g_{a_1}(r) \ge g_{a_2}(r)$ for some $r : 0 < r < 1$. Then since $g_{a_1}(r) < g_{a_2}(r)$ near $r = 1$, there exists r_0 such that $g_{a_1}(r_0) = g_{a_2}(r_0)$. This g_{a_1} and r_0 satisfy the condition of Theorem 1 with $\alpha = a_2$, $r(z) = r_0$, and $u = g_{a_1}$, but $g_{a_1} \notin X_{\lambda_{a_2}}$. This is a contradiction.

(2) Let $\beta = b + ic$, $c \neq 0$, be fixed. Let $\mathscr{B}(r) = g_\beta(r)/g_b(r)$, $0 \le r < 1$, and let $R = \{(a, r) : \frac{1}{2} < a < b \text{ and } 0 < r < 1\}$. Define

$$G(a; r) = g_a(r)/g_b(r), \qquad (a, r) \in \overline{R}.$$

Since $g_b(r) = (1-r^2)^{n(1-b)}\mathscr{F}(b : r)$, $\mathscr{F}(b : 0) = 1$, and $\mathscr{F}(a : r)$ is increasing, it follows that $g_b(r)$ cannot be zero and that $G(a; r)$ is well defined continuous

in each variable. Note also, by (1), that $G(a; r)$ is strictly increasing in the first variable. Let Ω be the simply connected open region surrounded by

$$\{G(\tfrac{1}{2}; r) : 0 \le r \le 1\} \cup \{(a, 1) : 0 \le a \le 1\} \cup \{(1, r) : 0 \le r \le 1\}.$$

Then

$$\Omega \subset \bigcup_{(a;r)\in R} (r, G(a; r)); \tag{11}$$

because if $(x, y) \in \Omega$ then $G(\frac{1}{2}; x) < y < G(b; x) = 1$, and by use of the intermediate value theorem we can find $a : \frac{1}{2} < a < b$ such that $G(a; x) = y$.

Next let $\mathscr{B} = \lim_{r\to 1} |\mathscr{B}(r)|$. We claim

$$0 < \mathscr{B} < 1. \tag{12}$$

It is obvious that $\mathscr{B} \le 1$. Also, since $|\mathscr{F}(\beta : r)| \le \mathscr{F}(b : r) \uparrow \mathscr{F}(b)$, by the dominated convergence theorem and (3),

$$|\mathscr{B}(1)| = \lim_{r\to 1} \frac{|g_\beta(r)|}{g_b(r)} = \frac{\lim_{r\to 1} |\mathscr{F}(\beta : r)|}{\lim_{r\to 1} \mathscr{F}(b : r)} = \frac{|\mathscr{F}(\beta : 1)|}{\mathscr{F}(b)}.$$

Since

$$\begin{aligned}\mathscr{F}(\beta : 1) &= \lim_{r\to 1} \mathscr{F}(\beta : r) = \lim_{r\to 1} F(n - n\beta, n - n\beta, n : r^2)\\ &= \frac{\Gamma(n)\Gamma(2n\beta - n)}{\Gamma^2(n\beta)},\end{aligned}$$

where $F(\cdots)$ is the Gaussian hypergeometric function (see [3] or [4, p. 54]), we have $\mathscr{B} \ne 0$. Suppose $|\mathscr{F}(\beta : 1)| = \mathscr{F}(b)$. Then $\gamma\mathscr{F}(\beta : 1) = \mathscr{F}(b)$ for some γ with $|\gamma| = 1$, that is,

$$\int_S \left(\frac{1}{|1 - \zeta_1|^{2n(1-b)}} - \frac{\gamma}{|1 - \zeta_1|^{2n(1-\beta)}} \right) d\sigma(\zeta) = 0,$$

from which follows $1 = \mathrm{Re}(\gamma|1 - \zeta_1|^{i2nc})$ a.e., $\zeta \in S$. This is impossible.

Now, since $\mathscr{F}(\beta : r)/\mathscr{F}(b : r)$ has a nonzero finite limit as $r \to 1$,

$$\mathscr{B}(r) = (1 - r^2)^{-inc} \frac{\mathscr{F}(\beta : r)}{\mathscr{F}(b : r)}$$

makes a curve that winds the origin infinitely many times as r approaches to 1. In particular, we can take, by (12), a positive sequence $\mathscr{B}(r_j)$ such that $1 > \mathscr{B}(r_j) \to \mathscr{B} > 0$ as $r_j \to 1$. Recalling (11), we can conclude that there are infinitely many r_j such that

$$(r_j, \mathscr{B}(r_j)) \subset \Omega \subset \bigcup (r, G(a; r)).$$

Therefore there are a and r_j such that $\mathscr{B}(r_j) = G(a; r_j)$, that is, $g_\beta(r_j) = g_a(r_j)$.

Proof of Theorem 3. That (2) $\Rightarrow$ (1) follows from Theorem 1. For the converse, first fix a nonreal β with $\mathrm{Re}\,\beta > \frac{1}{2}$. Take a and r as in Lemma 2(2). Then by (ii),

$$g_a(z)(1 - |z|^2)^{n(\beta-1)} \in C(\overline{B}),$$

and by (iii) and (2),

$$g_a(z) g_\beta(r) = \int_S g_a(\phi_z(r\zeta))\, d\sigma(\zeta),$$

but $g_a(z) \in X_{\lambda_a}$ (so that $g_a(z) \notin X_{\lambda_\beta}$). Therefore $1\mathrm{RP}(\beta; g_a)$ is false.

5. The case $\alpha = \frac{1}{2}$

When $\alpha = \frac{1}{2}$, we have a stronger result as expected in the corollary to Theorem 1.

Theorem 4. *Suppose that*

$$u(z) = (1 - |z|^2)^{n/2} \frac{1}{|z|^2} \log \frac{1}{1 - |z|^2} F(z)$$

for some $F \in C(\overline{B})$. *If for every* $z \in B$ *there corresponds an* $r(z) : 0 < r(z) < 1$ *and an automorphism* ψ_z *such that* (4) *holds, then*

$$u(z) = \frac{1}{\mathscr{G}(1/2)} \int_S P^{1/2}(z, \zeta) F(\zeta)\, d\sigma(\zeta), \qquad z \in B.$$

In particular, $1\mathrm{RP}(\frac{1}{2}; u)$ *is true.*

Proof of Theorem 4. The idea is easier than, but almost the same as, that of the proof of Theorem 1. Let $F'(z) = F(z)/\mathscr{G}(\frac{1}{2} : |z|)$. Then by Lemma 1 $F' \in C(\overline{B})$ also. Let

$$v(z) = \int_S P^{1/2}(z, \zeta) F'(\zeta)\, d\sigma(\zeta),$$

and

$$G(z) = \begin{cases} g_{1/2}(z)^{-1} v(z), & z \in B, \\ F'(z), & z \in S. \end{cases}$$

Then since $(1 - r^2)^{n/2} g_{1/2}(r)^{-1} \to 0$ as $r \to 1$, it follows that $G(r\zeta) \to G(\zeta)$ uniformly on $\zeta \in S$ as $r \to 1$. Hence $G \in C(\overline{B})$. Now let $h(z) = u(z) - v(z)$. It suffices to show that the function

$$H(z) = \begin{cases} g_{1/2}(z)^{-1} h(z), & z \in B, \\ 0, & z \in S, \end{cases}$$

is identically zero in B. $H \in C(\overline{B})$ by Lemma 1. Suppose $|H(z)| > 0$ for some $z \in B$. Let E be the set on which $|H(z)|$ takes its maximum, and let $z_0 \in E$ be such that $\operatorname{dist}(z_0, S) = \operatorname{dist}(E, S)$. Then by (4)

$$\begin{aligned} H(z) g_{1/2}(r(z)) &= \int_S g_{1/2}(z)^{-1} h \circ \psi_z(r(z)\zeta)\, d\sigma(\zeta) \\ &= \int_S \frac{g_{1/2}(\psi_z(r\zeta))}{g_{1/2}(z)} H \circ \psi_z(r\zeta)\, d\sigma(\zeta). \end{aligned}$$

If we apply z_0 with its corresponding $r(z_0) = r_0$, we obtain, by (2) and by the location of z_0,

$$|H(z_0)| g_{1/2}(r_0) < |H(z_0)| g_{1/2}(r_0),$$

which is a contradiction.

The growth condition in Theorem 4 is sharp in the sense that there is $u(z)$ such that $F(z)$ is bounded on B and $1\mathrm{RP}(\frac{1}{2}; u)$ is false. We can see this by following exactly the same lines of the proof of Theorem 2.

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