ONE RADIUS THEOREM FOR THE EIGENFUNCTIONS OF THE INVARIANT LAPLACIAN

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Abstract. Let $B$ be the open unit ball in $\mathbb{C}^n$ with its boundary $S$. Suppose that $\alpha \geq \frac{1}{2}$ and $u(z) = (1 - |z|^2)^{\alpha(1 - \alpha)} F(z)$ for some $F(z) \in C(\overline{B})$. If for every $z \in B$ there corresponds an $r(z) : 0 < r(z) < 1$ and an automorphism $\psi_z$ with $\psi_z(0) = z$ such that

$$u(z) = \frac{1}{g_\alpha(r(z))} \int_S u \circ \psi_z(r(z)\zeta) \, d\sigma(\zeta),$$

then $\tilde{\Delta} u(z) = -4n^2 (1 - \alpha) u(z)$, $z \in B$. Here $\tilde{\Delta}$ is the invariant Laplacian and $g_\alpha(r)$ is the hypergeometric function $F(n - n\alpha, n - n\alpha, n; r^2)$.

1. Introduction

Let $\psi_z$ be an automorphism, that is, a one-to-one holomorphic map onto itself, of the open unit ball $B$ of $\mathbb{C}^n$ satisfying $\psi_z(0) = z$. Let $\phi_z$ be one of such an automorphism defined by

$$\phi_z(w) = \frac{z - P_z w - s_z(w - P_z w)}{1 - \langle w, z \rangle}, \quad w \in B,$$

where $P_z$ is the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by $z$, $\langle w, z \rangle = \sum_{j=1}^n w_j z_j$, $s_z = \sqrt{1 - |z|^2}$, and $|z|^2 = \langle z, z \rangle$. For $u \in C^2(B)$, its invariant Laplacian $\tilde{\Delta}$ is well defined by

$$(\tilde{\Delta} u)(z) = \Delta(u \circ \psi_z)(0), \quad z \in B,$$

where $\Delta$ is the ordinary Laplacian. Our starting point is the following mean value theorem [4, Theorem 4.2.4].

Theorem A. Given a complex number $\lambda$, any function $u(z) \in C^2(B)$ satisfying
\begin{equation}
\tilde{\Delta} u(z) = \lambda u(z), \quad z \in B,
\end{equation}
has mean value property
\begin{equation}
g_\alpha(r) u(z) = \int_S u \circ \psi_z(r\zeta) \, d\sigma(\zeta), \quad z \in B, \quad 0 < r < 1.
\end{equation}
Here and throughout $\lambda$ and $\alpha$ are related to be
\[ \lambda = \lambda_\alpha = -4n^2\alpha(1 - \alpha), \]
\[ d\sigma \] is the normalized rotation invariant positive Borel measure on the boundary $S$ of $B$, and
\[ g_\alpha(r) = \int_S \left( \frac{1 - r^2}{|1 - r_\zeta|^{2\alpha}} \right) d\sigma(\zeta). \]
The space of all $u(z)$ satisfying (1) is denoted by $X_\lambda$. This space is invariant under automorphism. See [4, Chapter 4] for the terminology and related properties.

When $\lambda = 0$, Theorem A has a strong converse known as the one radius theorem [4, Theorem 4.3.4]. See [2] also for the same vein. Motivated by these, our goal here is in the description of the complex numbers $\alpha$ and the smoothness of functions $u(z)$ that makes the following property valid.

**One-radius property for $\alpha$ and $u$.** If for every $z \in B$ there corresponds an $r(z) : 0 < r(z) < 1$ such that
\[ g_\alpha(r(z))u(z) = \int_S u \circ \phi_z(r(z)\zeta) \ d\sigma(\zeta), \]
then $u(z) \in X_\lambda$.

We denote this property by $1\text{RP}(\alpha; u)$ in short. Rudin's one radius theorem says that $1\text{RP}(0; u) = 1\text{RP}(1; u)$ is true if $u(z) \in C(\overline{B})$. Izuchi [1] proved that $1\text{RP}(1; u)$ fails for a bounded real analytic $u(z)$. In this paper, we first confine ourselves to $\Re \alpha > \frac{1}{2}$. Our results say that under our growth condition on $u$, $1\text{RP}(\alpha; u)$ is true if and only if $\alpha$ is real (Theorem 3), and the growth condition on $u$ cannot be improvable (Theorem 2). Our last section deals with the case $\alpha = \frac{1}{2}$. All unexplained notations and properties will be referred to [4].

2. **One radius theorem for $X_\lambda$**

**Lemma 1.** Let $\alpha \in \mathbb{C}$. Then
\[ g_\alpha(r) = g_{1-\alpha}(r) = (1 - r^2)^{n(1-\alpha)} \mathcal{F}(\alpha : r), \]
where
\[ \mathcal{F}(\alpha : r) = \int_S \frac{(1 - r_\zeta)^{2n\alpha - n}}{|1 - r_\zeta|^{2n\alpha}} \ d\sigma(\zeta). \]
If $\alpha$ is real then $\mathcal{F}(\alpha : r)$ is an increasing function of $r$. If $\alpha > \frac{1}{2}$ then $\mathcal{F}(\alpha : r)$ tends to a finite limit $\mathcal{F}(\alpha) = \lim_{r \to 1} \mathcal{F}(\alpha : r)$. Also
\[ \mathcal{F}\left(\frac{1}{2} : r\right) = \frac{1}{r^2} \log \frac{1}{1 - r^2} \mathcal{F}\left(\frac{1}{2} : r\right)^{-1} \]
is a continuous function of $r$ that tends to a finite limit $\mathcal{F}(\frac{1}{2}) = \lim_{r \to 1} \mathcal{F}(\frac{1}{2} : r)$.

**Theorem 1.** Suppose $\alpha > \frac{1}{2}$ and $u(z) = (1 - |z|^2)^{n(1-\alpha)}F(z)$ for some $F(z) \in C(\overline{B})$. If for every $z \in B$ there corresponds an $r(z) : 0 < r(z) < 1$ and an
automorphism $\psi_z$ such that

$$(4) \quad g_a(r(z))u(z) = \int_S u \circ \psi_z(r(z)\zeta) \, d\sigma(\zeta),$$

then

$$u(z) = \frac{1}{\mathcal{F}(\alpha)} \int_S P^\alpha(z, \zeta)F(\zeta) \, d\sigma(\zeta), \quad z \in B,$$

where $P(z, w)$ denotes the invariant Poisson kernel. In particular, $1\mathcal{R}(\alpha; u)$ is true.

Proof of Lemma 1. By [4, Remark, p. 44],

$$\mathcal{F}(\alpha : r) = \int_S \frac{(1 - r^2)^{2n\alpha - n}}{|1 - \langle r\eta, \phi_r(\zeta) \rangle|^{2n\alpha}} P(\phi_r^{-1}(0), \zeta) \, d\sigma(\zeta)$$

$$= \int_S \frac{(1 - r^2)^{2n\alpha - n}}{|1 - \langle \phi_r(0), \phi_r(\zeta) \rangle|^{2n\alpha}} \frac{(1 - r^2)^n}{|1 - \langle \eta, \zeta \rangle|^{2n}} \, d\sigma(\zeta),$$

which turns out to be

$$\int_S \frac{d\sigma(\zeta)}{|1 - \langle \eta, \zeta \rangle|^{2n(1-\alpha)}}$$

once we use the identity [4, Theorem 2.2.5]

$$(5) \quad 1 - \langle \psi_a(z), \psi_a(w) \rangle = \frac{(1 - |a|^2)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)},$$

$a \in B$, $z, w \in \overline{B}$. Hence $(1 - r^2)^n(1 - \alpha)\mathcal{F}(\alpha : r) = g_{1-\alpha}(r)$. That $g_a(r) = g_{1-\alpha}(r)$ is noted in [4, Corollary, p. 51]. The second part follows from the subharmonicity of the integrand and [4, Proposition 1.4.10].

Proof of Theorem 1. Let

$$v(z) = \frac{1}{\mathcal{F}(\alpha)} \int_S P^\alpha(z, \zeta)F(\zeta) \, d\sigma(\zeta), \quad z \in B.$$ 

Then $v(z) \in X_\lambda$ because $P^\alpha(z, \zeta) \in X_\lambda$ [4, Theorem 4.2.2], so that it suffices to show that $h(z) \equiv u(z) - v(z) \equiv 0$. Consider first the function defined by

$$G(z) = \begin{cases} (1 - |z|^2)^{\alpha - 1}v(z), & z \in B, \\ F(z), & z \in S. \end{cases}$$

For $\zeta \in S$,

$$|G(r\zeta) - G(\zeta)| \leq |G(r\zeta) - \mathcal{F}(\alpha : r) F(\zeta)| + |F(\zeta)| \left| \frac{\mathcal{F}(\alpha : r)}{\mathcal{F}(\alpha)} - 1 \right|$$

$$= |(I)| + |(II)|.$$ 

Since $F(z) \in C(S)$, by Lemma 1, $(II) \to 0$ as $r \to 1$ uniformly on $S$. Also since

$$(I) = \frac{1}{\mathcal{F}(\alpha)} \int_S \frac{(1 - r^2)^{2n\alpha - n}}{|1 - \langle r\zeta, \eta \rangle|^{2n\alpha}} [F(\eta) - F(\zeta)] \, d\sigma(\eta),$$

by dividing $S$ into two parts—a small neighborhood $Q$ of $\zeta$ and $S - Q$—it is routine to see that $(I) \to 0$ as $r \to 1$ uniformly on $S$. Therefore $G(z) \in C(\overline{B})$. 
Now the function $H(z)$ defined by

$$H(z) = \begin{cases} (1 - |z|^2)^{n(\alpha - 1)} h(z), & z \in B, \\ 0, & z \in S, \end{cases}$$

is of $C(\overline{B})$. We will show $H \equiv 0$ in $B$. Suppose $|H(z)| > 0$ for some $z \in B$. Then the set $E$ on which $|H(z)|$ takes the maximum is compact. We can take $z_0 \in E$ such that $\text{dist}(z_0, S) = \text{dist}(E, S)$. Since both $u(z)$ and $v(z)$ satisfy (4), we have (4) with $h(z)$ in place of $u(z)$, so that

$$H(z)g_\alpha(r(z)) = \int_S (1 - |z|^2)^{n(\alpha - 1)} h \circ \psi_z(r(z)\zeta) \, d\sigma(\zeta), \quad z \in B.$$ 

But the right side of (6) is

$$\int_S P^{1-\alpha}(r(z)\zeta, z) H \circ \psi_z(r(z)\zeta) \, d\sigma(\zeta)$$

once we use the identity (5). If we apply (6) and (7) to $z_0$ with its corresponding $r(z_0) = r_0$, we obtain, by (3),

$$|H(z_0)| = \frac{1}{\mathcal{F}(\alpha : r_0)} \left| \int_S \frac{H \circ \psi_{z_0}(r_0\zeta)}{|1 - \langle r_0 z_0, \zeta \rangle|^{n(1 - \alpha)}} \, d\sigma(\zeta) \right|,$$

which is, by (3) once more, strictly less than

$$\frac{\mathcal{F}(\alpha : r_0)|z_0|}{\mathcal{F}(\alpha : r_0)} |H(z_0)| \leq |H(z_0)|.$$

This contradiction proves our assertion.

### 3. On smoothness

Because of the symmetry, $g_\alpha = g_{1-\alpha}$ and $X_{\lambda_{\alpha}} = X_{\lambda_{1-\alpha}}$, we can have the proper substitute for the case $\alpha < \frac{1}{2}$, that is, Theorem 1 with all $\alpha$ replaced by $1 - \alpha$. On the other hand, if $\alpha < \frac{1}{2}$ and $(1 - |z|^2)^{n(\alpha - 1)} u(z)$ can be of $C(\overline{B})$, then we can easily check from Theorem 1 that $1 \text{RP}(\alpha; u) = 1 \text{RP}(1 - \alpha; u)$ is true. But if $\alpha \leq \frac{1}{2}$, there is no $u \in X_{\alpha}, u \neq 0$ with $(1 - |z|^2)^{n(\alpha - 1)} u(z)$ bounded. This can be checked by comparing the order of $g_\alpha(r)$ as $r \to 1$ after integrating $(1 - |\psi_z(r\zeta)|^2)^{n(\alpha - 1)} u \circ \psi_z(r\zeta)$ with respect to $d\sigma(\zeta)$ and using (2) in Theorem A and (5). This fact and Theorem 1 gives the following solution to the boundary value problem.

**Corollary.** Let $\alpha$ be real and let $f \in C(S), f \neq 0$, be given. Then there is a solution $u(z)$ such that

(i) $\tilde{\Delta} u(z) = \lambda u(z)$ in $B$,

(ii) with boundary value $f(\zeta)$, $(1 - |z|^2)^{n(\alpha - 1)} u(z)$ is of $C(\overline{B})$

if and only if $\alpha > \frac{1}{2}$. In fact,

$$u(z) = \frac{1}{\mathcal{F}(\alpha)} \int_S P^\alpha(z, \zeta) f(\zeta) \, d\sigma(\zeta)$$

is the unique solution when $\alpha > \frac{1}{2}$.

Our proof of Theorem 1 covers the case $\alpha = 1$. But in (8) equality occurs only when $\alpha = 1$ because $\mathcal{F}(\alpha : r)$ is strictly increasing if $\alpha$ is real and $\alpha \neq 1$. 

So it seems to be that the compactness of $E$ when $\alpha \neq 1$ is not so much crucial compared to the case $\alpha = 1$. Nonetheless we can explain by use of an example of Izuchi that our growth condition cannot be weakened.

**Theorem 2.** For $\alpha > \frac{1}{2}$, there is a function $u(z)$ such that

(i) $u(z) \notin X_{\lambda}$,

(ii) $(1 - |z|^2)^{n(\alpha-1)}u(z)$ is bounded real analytic in $B$,

(iii) for every $z \in B$, there is a radius $r(z) : 0 < r(z) < 1$ by which $u(z)$ satisfies (4).

**Proof.** Izuchi [1] constructed a bounded positive radial real analytic function $F(z)$ on $B$ such that

\begin{equation}
\sup_{\zeta \in S} F \circ \phi_z(\delta_j \zeta) < F(z) < \inf_{\zeta \in S} F \circ \phi_z(\sigma_j \zeta)
\end{equation}

for some $\delta_j, \sigma_j$ depend on $z$ (see [1, p. 830]). If we fix $\alpha$ and set $u(z) = g_{\alpha}(z)F(z)$, then by (9) and (2),

\begin{equation}
\int_{S} u \circ \phi_z(\delta_j \zeta) d\sigma(\zeta) < F(z) \int_{S} g_{\alpha} \circ \phi_z(\delta_j \zeta) d\sigma(\zeta) = u(z)g_{\alpha}(\delta_j)
\end{equation}

and

\begin{equation}
u(z)g_{\alpha}(\sigma_j) = F(z) \int_{S} g_{\alpha} \circ \phi_z(\sigma_j \zeta) d\sigma(\zeta) < \int_{S} u \circ \phi_z(\sigma_j \zeta) d\sigma(\zeta),
\end{equation}

so that by the continuity on $r$ of

\begin{equation}
\frac{1}{g_{\alpha}(r)} \int_{S} u \circ \phi_z(r \zeta) d\sigma(\zeta),
\end{equation}

(iii) is valid. This $u(z)$ satisfy (i), because of the strict inequality in (10), and (ii) also, because $F(z)$ and $(1 - |z|^2)^{n(\alpha-1)}g_{\alpha}(z)$ are bounded real analytic.

4. **One-radius property fails for nonreal $\alpha$**

**Lemma 2.** (1) If $\frac{1}{2} < a_1 < a_2$ then $g_{a_1}(r) < g_{a_2}(r), 0 < r < 1$.

(2) For a given nonreal $\beta$ with $\text{Re} \beta > \frac{1}{2}$, there exist $r$ and $a$ such that

(i) $0 < r < 1$, (ii) $\frac{1}{2} < a < \text{Re} \beta$, and (iii) $g_a(r) = g_{\beta}(r)$.

**Theorem 3.** Let $\text{Re} \alpha > \frac{1}{2}$. Then the following are equivalent:

(1) $1\text{RP}(\alpha ; u)$ is true for all $u(z)$ with $(1 - |z|^2)^{n(\alpha-1)}u(z) \in C(B)$.

(2) $\alpha$ is real.

**Proof of Lemma 2.** (1) Suppose $g_{a_1}(r) \geq g_{a_2}(r)$ for some $r : 0 < r < 1$. Then since $g_{a_1}(r) < g_{a_2}(r)$ near $r = 1$, there exists $r_0$ such that $g_{a_1}(r_0) = g_{a_2}(r_0)$. This $g_{a_1}$ and $r_0$ satisfy the condition of Theorem 1 with $\alpha = a_2$, $r(z) = r_0$, and $u = g_{a_1}$, but $g_{a_1} \notin X_{\lambda_a}$. This is a contradiction.

(2) Let $\beta = b + ic$, $c \neq 0$, be fixed. Let $G(r) = g_{\beta}(r)/g_{b}(r), 0 \leq r < 1$, and let $R = \{(a, r) : \frac{1}{2} < a < b$ and $0 < r < 1\}$. Define

\[ G(a ; r) = g_a(r)/g_{b}(r), \quad (a, r) \in R. \]

Since $g_{b}(r) = (1 - r^2)^{n(1-b)}F(b : r)$, $F(b : 0) = 1$, and $F(a : r)$ is increasing, it follows that $g_{b}(r)$ cannot be zero and that $G(a ; r)$ is well defined continuous.
in each variable. Note also, by (1), that \(G(a; r)\) is strictly increasing in the first variable. Let \(\Omega\) be the simply connected open region surrounded by

\[
\{G(\frac{1}{2}; r) : 0 \leq r \leq 1\} \cup \{(a, 1) : 0 \leq a \leq 1\} \cup \{(1, r) : 0 \leq r \leq 1\}.
\]

Then

\[
\Omega \subset \bigcup_{(a, r) \in R} (r, G(a; r));
\]

because if \((x, y) \in \Omega\) then \(G(\frac{1}{2}; x) < y < G(b; x) = 1\), and by use of the intermediate value theorem we can find \(a : \frac{1}{2} < a < b\) such that \(G(a; x) = y\).

Next let \(\mathcal{B} = \lim_{r \to 1} |\mathcal{B}(r)|.\) We claim

\[
|\mathcal{B}(1)| = \lim_{r \to 1} \frac{|g_\beta(r)|}{g_\beta(r)} = \lim_{r \to 1} \frac{\mathcal{F}(\beta : r)}{\mathcal{F}(\beta : r)} = |\mathcal{F}(\beta : 1)|.
\]

Since

\[
\mathcal{F}(\beta : 1) = \lim_{r \to 1} \mathcal{F}(\beta : r) = \lim_{r \to 1} F(n - n\beta, n - n\beta, n : r^2) = \frac{\Gamma(n)\Gamma(2n\beta - n)}{\Gamma^2(n\beta)},
\]

where \(F(\cdots)\) is the Gaussian hypergeometric function (see [3] or [4, p. 54]), we have \(\mathcal{B} \neq 0\). Suppose \(|\mathcal{F}(\beta : 1)| = \mathcal{F}(b).\) Then \(\gamma\mathcal{F}(\beta : 1) = \mathcal{F}(b)\) for some \(\gamma\) with \(|\gamma| = 1\), that is,

\[
\int_S \left(\frac{1}{|1 - \zeta_1|^{2n(1-b)}} - \frac{\gamma}{|1 - \zeta_1|^{2n(1-b)}}\right) d\sigma(\zeta) = 0,
\]

from which follows \(1 = \Re(\gamma|1 - \zeta_1|^{2n})\) a.e., \(\zeta \in S\). This is impossible.

Now, since \(\mathcal{F}(\beta : r)/\mathcal{F}(b : r)\) has a nonzero finite limit as \(r \to 1\),

\[
\mathcal{B}(r) = (1 - r^2)^{-inc}\frac{\mathcal{F}(\beta : r)}{\mathcal{F}(b : r)}
\]

makes a curve that winds the origin infinitely many times as \(r\) approaches to 1. In particular, we can take, by (12), a positive sequence \(\mathcal{B}(r_j)\) such that

\[
1 > \mathcal{B}(r_j) \to \mathcal{B} > 0 \quad \text{as} \quad r_j \to 1.
\]

Recalling (11), we can conclude that there are infinitely many \(r_j\) such that

\[
(r_j, \mathcal{B}(r_j)) \subset \Omega \subset \bigcup (r, G(a; r)).
\]

Therefore there are \(a\) and \(r_j\) such that \(\mathcal{B}(r_j) = G(a; r_j)\), that is, \(g_\beta(r_j) = g_a(r_j)\).

**Proof of Theorem 3.** That (2) \(\Rightarrow\) (1) follows from Theorem 1. For the converse, first fix a nonreal \(\beta\) with \(\Re \beta > \frac{1}{2}\). Take \(a\) and \(r\) as in Lemma 2(2). Then by (ii),

\[
g_a(z)(1 - |z|^2)^{n(\beta - 1)} \in C(\mathcal{B}),
\]

and by (iii) and (2),

\[
g_a(z)g_\beta(r) = \int_S g_a(\phi_z(r_\zeta)) d\sigma(\zeta),
\]

but \(g_a(z) \in X_{\lambda_a}\) (so that \(g_a(z) \notin X_{\lambda_\beta}\)). Therefore \(1\)RP(\(\beta\); \(g_a\)) is false.
5. The case $\alpha = \frac{1}{2}$

When $\alpha = \frac{1}{2}$, we have a stronger result as expected in the corollary to Theorem 1.

**Theorem 4.** Suppose that

$$u(z) = (1 - |z|^2)^{n/2} \frac{1}{|z|^2} \log \frac{1}{1 - |z|^2} F(z)$$

for some $F \in C(\overline{B})$. If for every $z \in B$ there corresponds an $r(z) : 0 < r(z) < 1$ and an automorphism $\psi_z$ such that (4) holds, then

$$u(z) = \frac{1}{\mathcal{G}(1/2)} \int_S P^{1/2}(z, \zeta) F(\zeta) \, d\sigma(\zeta), \quad z \in B.$$

In particular, $1RP(\frac{1}{2}; u)$ is true.

**Proof of Theorem 4.** The idea is easier than, but almost the same as, that of the proof of Theorem 1. Let $F'(z) = F(z)/\mathcal{G}(\frac{1}{2} : |z|)$. Then by Lemma 1 $F' \in C(\overline{B})$ also. Let

$$v(z) = \int_S P^{1/2}(z, \zeta) F'(\zeta) \, d\sigma(\zeta),$$

and

$$G(z) = \begin{cases} g_{1/2}(z)^{-1} v(z), & z \in B, \\ F'(z), & z \in S. \end{cases}$$

Then since $(1 - r^2)^{n/2} g_{1/2}(r)^{-1} \to 0$ as $r \to 1$, it follows that $G(r\zeta) \to G(\zeta)$ uniformly on $\zeta \in S$ as $r \to 1$. Hence $G \in C(\overline{B})$. Now let $h(z) = u(z) - v(z)$. It suffices to show that the function

$$H(z) = \begin{cases} g_{1/2}(z)^{-1} h(z), & z \in B, \\ 0, & z \in S, \end{cases}$$

is identically zero in $B$. $H \in C(\overline{B})$ by Lemma 1. Suppose $|H(z)| > 0$ for some $z \in B$. Let $E$ be the set on which $|H(z)|$ takes its maximum, and let $z_0 \in E$ be such that $\text{dist}(z_0, S) = \text{dist}(E, S)$. Then by (4)

$$H(z) g_{1/2}(r(z)) = \int_S g_{1/2}(z)^{-1} h \circ \psi_z(r(z) \zeta) \, d\sigma(\zeta)$$

$$= \int_S \frac{g_{1/2}(\psi_z(r(\zeta)))}{g_{1/2}(z)} H \circ \psi_z(r(\zeta)) \, d\sigma(\zeta).$$

If we apply $z_0$ with its corresponding $r(z_0) = r_0$, we obtain, by (2) and by the location of $z_0$,

$$|H(z_0)| g_{1/2}(r_0) < |H(z_0)| g_{1/2}(r_0),$$

which is a contradiction.
The growth condition in Theorem 4 is sharp in the sense that there is \( u(z) \) such that \( F(z) \) is bounded on \( B \) and \( 1\text{RP}(\frac{1}{2}; u) \) is false. We can see this by following exactly the same lines of the proof of Theorem 2.

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References


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