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ON THE CARTESIAN PRODUCTS OF LINDELÖF SPACES WITH ONE FACTOR HEREDITARILY LINDELÖF

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ABSTRACT. E. Michael asked the following question: Is there a space X such that $Y \times X$ is Lindelöf for every hereditarily Lindelöf space Y but X^2 is not. The aim of this paper is to present a construction that provides such an example.

In this paper we construct for every $n \in N$ a space X_n such that for every hereditarily Lindelöf space Y, $Y \times X_n^n$ is Lindelöf and X_n^{n+1} is not. Moreover, we also obtain, using the same technique, a space X_ω such that for every $n \in N$ and every hereditarily Lindelöf space Y, $Y \times X_\omega^n$ is Lindelöf but X_ω^ω is not.

Let us recall that another example of X_ω was presented in [A2, A3]. The present construction is much simpler than the previous one.

For related results and constructions see [A5, T, A6, A4].

Our topological terminology follows [E]. In particular, if M is a subspace of a topological space X then the symbol X_M stands for the set X with a new topology generated by $\mathscr{T} = \{U \subset X\colon U \text{ is open in } X\} \cup \{\{x\}\colon x \in X \backslash M\}$. In the sequel Q and N stand for the set of all rational and natural numbers respectively. Let us denote by $\mathscr{B}$ a countable base for Q consisting of open intervals. If $x \in Q$ then put $\mathscr{B}_x = \{B \in \mathscr{B}\colon x \in B\}$. The symbols ω and ω_1 denote the first infinite ordinal number and the first uncountable ordinal number, respectively.

The projection of Q^{ω_1} onto Q^T where $T \subset \omega_1$ will be denoted by P_T. If α is an ordinal number then we shall identify α with the set of its predecessors. If $\alpha < \beta$, and $x \in Q^\beta$ then let us denote $P_\alpha(x)$ by $x|\alpha$.

Example. For all $n \in N$ there is X_n such that for every hereditarily Lindelöf space Y, $Y \times (X_n)^n$ is Lindelöf but $(X_n)^{n+1}$ is not. Moreover, there is also X_ω such that for all $n \in N$, $Y \times (X_\omega)^n$ is Lindelöf but $(X_\omega)^\omega$ is not.

We now consider the proof of the example. There exists a family $\{A_\alpha\colon 1 \le \alpha < \omega_1\}$ such that

(1) A_α is a countable set consisting of strictly increasing sequences of rational numbers of length α for $1 \le \alpha < \omega_1$,

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(2) if $\alpha < \beta < \omega_1$ then $P_\alpha(A_\beta) = A_\alpha$,

(3) if $a = (a(\lambda))_{\lambda<\alpha} \in A_\alpha$ then a is a continuous function from α into Q and $\sup\{a(\lambda)\colon \lambda < \alpha\}$ is a rational number (see [J, p. 91], the construction of the Aronszajn tree).

Let us attach to $a \in A_\alpha$, for $1 \le \alpha < \omega_1$, $x_a \in Q^{\omega_1}$ such that

$$x_a(\beta) = \begin{cases} a(\beta) & \text{if } \beta < \alpha, \\ \sup\{a(\lambda)\colon \lambda \le \alpha\} & \text{if } \beta \ge \alpha. \end{cases}$$

Put $Z_\alpha = \{x \in Q^{\omega_1}$: there is a $a \in A_\alpha$ such that $x = x_a\}$. If $x \in \bigcup\{Z_\alpha\colon 1 \le \alpha < \omega_1\}$ then $\alpha(x)$ is such an ordinal number that $x \in Z_{\alpha(x)}$.

Let $\{S_i\colon i \le n+1\}$ be a cover of ω_1 consisting of pairwise disjoint stationary sets and put $K_i = \bigcup\{S_j\colon j \ne i\}$ for all $i \le n+1$.

If S is a subset of ω_1 then put $X(S) = \bigcup\{Z_\alpha\colon \alpha \in S\}$.

The desired space X_n is defined by $X_n = \bigoplus_{i\le n+1}(X(\omega_1)_{X(K_i)})$ where $X(\omega_1)$ and $X(K_i)$ are subspaces of Q^{ω_1}.

In case of X_ω we have to consider infinite partition $\{S_n\colon n \in N\}$ of ω_1 consisting of stationary sets and put

$$X_\omega = \bigoplus_{i=1}^{\infty}(X(\omega_1))_{X(K_i)}, \quad \text{where } K_i = \bigcup\{S_j\colon j \ne i\}.$$

Since $D = \{x = (x_i)_{i=1}^{n+1} \in \prod_{i=1}^{n+1}(X(\omega_1)_{X(K_i)})$: the coordinates of x are equal$\}$ is a discrete closed and uncountable subset of $(X_n)^{n+1}$, we infer that $(X_n)^{n+1}$ is not Lindelöf.

In order to prove that $Y \times (X_n)^n$ is Lindelöf for every hereditarily Lindelöf space Y we shall need

Lemma 1. *If S is a stationary subset of ω_1 then $Y \times (X(S))^n$ is Lindelöf for every hereditarily Lindelöf space Y.*

Proof. Let Y be a hereditarily Lindelöf space and $\mathcal{U}$ an open cover of $Y \times (X(S))^n$. If $x = x_a \in Z_\alpha$, where $a \in A_\alpha$, $x_a(\alpha) \in B \in \mathcal{B}$, and $\alpha < \lambda$, then put

$$F(x, B, \lambda) = \left(\prod_{\beta<\omega_1} F_\beta\right) \cap X(\omega_1),$$

where

$$F_\beta = \begin{cases} \{x(\beta)\} & \text{if } \beta \le \alpha, \\ B & \text{if } \beta = \lambda, \\ Q & \text{otherwise.} \end{cases}$$

For $x = (x_1, \dots, x_n) \in (X(S))^n$ and $\prod_{i=1}^n B_i \in (\mathcal{B})^n$, where $x_i \in Z_{\alpha(x_i)}$ and $x_i(\alpha(x_i)) \in B_i$ for all $i \le n$ put $A(x, \prod_{i=1}^n B_i) = \{y \in Y\colon \exists$ open H_y, $\sup\{\alpha(x_i)\colon i \le n\} < \lambda(y) < \omega_1$, and $U \in \mathcal{U}$ such that $y \in H_y$ and $H_y \times \prod_{i=1}^n(F(x_i, B_i, \lambda(y))) \subset U\}$.

We can assume, without loss of generality, that $\lambda(y)$ for $y \in A(x, \prod_{i=1}^n B_i)$ is as small as possible. Since $\{y \in A(x, \prod_{i=1}^n B_i)\colon \lambda(y) \le \beta\}$ is open for all $\beta < \omega_1$ and $A(x, \prod_{i=1}^n B_i)$ is Lindelöf as a subspace of Y, we infer that if

$A(x, \prod_{i=1}^n B_i) \neq \varnothing$ then $\sup\{\lambda(y): y \in A(x, \prod_{i=1}^n B_i)\}$ is less than ω_1. Put

$$\lambda\left(x, \prod_{i=1}^n B_i\right) = \begin{cases} \sup\{\lambda(y): y \in A(x, \prod_{i=1}^n B_i)\} & \text{if } A(x, \prod_{i=1}^n B_i) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Using again the Lindelöf property of $A(x, \prod_{i=1}^n B_i)$ we can find $\{y_j: j \in N\} \subset A(x, \prod_{i=1}^n B_i)$ such that $\bigcup\{H_{y_j}: j \in N\} = A(x, \prod_{i=1}^n B_i)$, where H_{y_i} was defined in connection with $A(x, \prod_{i=1}^n B_i)$.

Let C be a subset of all countable ordinal numbers satisfying the following conditions:

(a) If $\alpha \in C$ then there is a sequence $(\alpha_n)_{n=1}^\infty$ in S converging to α; we do not require that $\alpha \in S$.

(b) If $\alpha \in C$ then for all $(\beta_1, \dots, \beta_n) \in (S \cap \alpha)^n$, $x = (x_1, \dots, x_n) \in \prod_{i=1}^n Z_{\beta_i}$ and $\prod_{i=1}^n B_i \in (\mathscr{B})^n$ where $(x_i(\beta_i))_{i=1}^n \in \prod_{i=1}^n B_i$, $\lambda(x, \prod_{i=1}^n B_i) < \alpha$.

In order to continue the proof of Lemma 1 we need

Claim 1. C is closed and unbounded in ω_1.

Proof. Observe that C is closed. Hence the proof will be finished if we show that C is unbounded. Fix $\beta < \omega_1$. There is $\beta_0 \in S \backslash \beta$. Put

$$\beta_1' = \sup\left\{\lambda\left(x, \prod_{i=1}^n B_i\right) : x = (x_i)_{i=1}^n \in \left(\bigcup\{Z_\alpha: \alpha \in S \cap (\beta_0 + 1)\}\right)^n, \prod_{i=1}^n B_i \in \prod_{i=1}^n \mathscr{B}_{x_i(\alpha(x_i))}\right\}$$

and choose $\beta_1 \in S \backslash \beta_1'$. If β_j is defined then put

$$\beta_{j+1}' = \sup\left\{\lambda\left(x, \prod_{i=1}^n B_i\right) : x = (x_i)_{i=1}^n \in \left(\bigcup\{Z_\alpha: \alpha \in S \cap (\beta_j + 1)\}\right)^n, \prod_{i=1}^n B_i \in \prod_{i=1}^n \mathscr{B}_{x_i(\alpha(x_i))}\right\}$$

and choose $\beta_{j+1} \in S \backslash \beta_{j+1}'$. Then $\gamma = \sup\{\beta_j: j \in N\} \in C$.

Since S is a stationary subset in ω_1 and C is closed and unbounded in ω_1, there is $\alpha \in S \cap C$. Put

$$\mathscr{D} = \left\{H_{y_i} \times \prod_{j=1}^n F(x_j, B_j, \lambda(y_j)): y_i \in A\left(x, \prod_{j=1}^n B_j\right), i \in N, \prod_{j=1}^n B_j \in \prod_{j=1}^n \mathscr{B}_{x_j}(\alpha(x_j)), \text{ and } x = (x_1, \dots, x_n) \in \left(\bigcup\{Z_\gamma: \gamma \in S \cap \alpha\}\right)^n\right\}.$$

Observe that $\mathscr{D}$ is a countable family that refines $\mathscr{U}$. In order to finish the proof of Lemma 1 it is enough to show that

Claim 2. $\bigcup \mathscr{D} = Y \times (X(S))^n$.

Proof. Let (y, x) be an arbitrary point of $Y \times (X(S))^n$. We shall consider two cases:

Case 1. $x \in (\bigcup\{Z_\gamma\colon \gamma \in S \cap (\alpha+1)\})^n$.

Case 2. Case 1 does not hold.

Proof of Case 1. There are an open subset $y \in H$ of Y, $U \in \mathscr{U}$, finite subsets T_i of ω_1, and $B_\gamma \in \mathscr{B}$ for all $\gamma \in T_i$ and $i \le n$ such that

$$(y, x) \in H \times \prod_{i=1}^n \left(P_{T_i}^{-1}\left(\prod_{\gamma \in T_i} B_\gamma\right)\right) \cap (X(S))^n \subset U.$$

Put $T_i = T_{i,1} \cup T_{i,2}$, where $T_{i,1} = T_i \cap \alpha(x_i)$ and $T_{i,2} = T_i \backslash T_{i,1}$ for all $i \le n$. Since the coordinates of x_i for $\gamma \ge \alpha(x_i)$ are the same, we can assume, without loss of generality, that $T_{i,2}$ is not empty and that there is $B_i \in \mathscr{B}$ such that $B_\gamma = B_i$ for all $\gamma \in T_{i,2}$ and $i \le n$.

The sequence $(x_i(\gamma))_{\gamma < \alpha(x_i)}$ converges to $x_i(\alpha(x_i))$ so there is $\sup T_{i,1} < \gamma_i \le \alpha(x_i)$ such that $\gamma_i \in S \cap \alpha$ and $x_i(\gamma_i) \in B_i$ for all $i \le n$. If $\lambda_i = \sup T_{i,2}$, $x_i = x_{a_i}$, and $\lambda = \sup\{\lambda_i\colon i = 1, 2, \dots, n\}$ then $(y, x) \in H \times \prod_{i=1}^n F(x_{a_i|\gamma_i}, B_i, \lambda)$. Since elements of $X(S)$ are increasing sequences and B_i are intervals for all $i \le n$, we infer that

$$\begin{aligned}(y, x) &\in H \times \prod_{i=1}^n F(x_{a_i|\gamma_i}, B_i, \lambda)\\ &\subset H \times \prod_{i=1}^n \left(p_{T_i}^{-1}\left(\prod_{\gamma \in T_i} B_\gamma\right)\right) \cap (X(S))^n \subset U.\end{aligned} \tag{4}$$

From the last fact it follows that

$$y \in A\left((x_{a_1|\gamma_1}, \dots, x_{a_n|\gamma_n}), \prod_{i=1}^n B_i\right). \tag{5}$$

Observe that since $\gamma_i < \alpha$ for all $i \le n$, we have

$$(x_{a_1|\gamma_1}, \dots, x_{a_n|\gamma_n}) \in \left(\bigcup\{Z_\gamma\colon \gamma \in S \cap \alpha\}\right)^n. \tag{6}$$

From (5) it follows that there is

$$y_k \in A\left((x_{a_1|\gamma_1}, \dots, x_{a_n|\gamma_n}), \prod_{i=1}^n B_i\right)$$

such that

$$(y, (x_{a_1|\gamma_1}, \dots, x_{a_n|\gamma_n})) \in H_{y_k} \times \prod_{i=1}^n F(x_{a_i|\gamma_i}, B_i, \lambda(y_k)). \tag{7}$$

Since $x_i|\gamma_i = x_{a_i|\gamma_i}|\gamma_i$ and for all $\gamma_i < \gamma < \omega_1$, $x_i(\gamma) \in B_i$ and $i \le n$, we infer by (7) that $(y, x) \in H_{y_k} \times \prod_{i=1}^n F(x_{a_i|\gamma_i}, B_i, \lambda(y_k))$. This completes the proof of Case 1.

Proof of Case 2. Let use assume that $x = (x_1, \dots, x_n)$ is an arbitrary point of $(X(S))^n$ such that $x_i = x_{a_i}$. Then let $z = (z_1, \dots, z_n)$ be such that

$$z_i = \begin{cases} x_i & \text{if } x_i \in U\{Z_\gamma\colon \gamma \in S \cap (\alpha+1)\}, \\ x_{a_i|\alpha} & \text{otherwise.}\end{cases}$$

Then from Case 1 it follows that there is $D = D_1 \times D_2 \in \mathscr{D}$ containing (y, z) where $D_1 \subset Y$, $D_2 = \prod_{i=1}^{n} D_{2,i}$, and $D_{2,i} \subset X(S)$ for all $i \le n$. Since $\alpha \in C$, we infer that

$$(8) \qquad P_\alpha^{-1} P_\alpha(D_{2,i}) = D_{2,i} \quad \text{for all } i \le n.$$

Since $z_i|\alpha = x_i$ for all $i \le n$, we conclude, applying (8), that $(y, x) \in D$. This completes the proof of Lemma 1.

Now we are in a position to prove

Lemma 2. *If S is a stationary subset of ω_1 then for every hereditarily Lindelöf space Y the product $Y \times (X(\omega_1)_{X(S)})^n$ is Lindelöf.*

Proof. Suppose that Y is a hereditarily Lindelöf space. We shall prove Lemma 3 by induction with respect to n. In order to simplify the induction we shall assume that $(X(\omega_1)_{X(S)})^0$ is a one point set. If $n = 0$ then there is nothing to prove. Let us suppose that $Y \times (X(\omega_1)_{X(S)})^n$ is Lindelöf and that $\mathscr{U}$ is an open cover of $Y \times (X(\omega_1)_{X(S)})^{n+1}$ consisting of the sets of the form $H \times \prod_{i=1}^{n+1} P_{T_i}^{-1}(\prod_{\gamma \in T_i} B_\gamma) \cap (X(\omega_1))^{n+1}$ where $B_\gamma \in \mathscr{B}$ for all $\gamma \in T_i$ and T_i is a finite subset of ω_1 for $i \le n+1$. Then by Lemma 1 there is a countable subfamily $\mathscr{U}'$ of $\mathscr{U}$ such that $Y \times (X(S))^{n+1} \subset \bigcup \mathscr{U}'$. Since $\mathscr{U}'$ is countable, there is $\alpha < \omega_1$ such that

$$(9) \qquad \begin{array}{l} \text{if } U = U_1 \times \prod_{i=1}^{n+1} U_{2,i} \in \mathscr{U}', \text{ where } U_1 \subset Y \text{ and } U_{2,i} \subset X(\omega_1) \\ \text{for all } i \le n+1, \text{ then } P_\alpha^{-1} P_\alpha(U_{2,i}) = U_{2,i}. \end{array}$$

Observe that from (9) it follows that

$$(10) \qquad \begin{array}{l} \text{if } y \in Y \text{ and } x = (x_1, \dots, x_{n+1}) \in \left(\bigcup \{Z_\gamma : \gamma \in \omega_1 \backslash \alpha\}\right)^{n+1}, \\ \text{then } (y, x) \in \bigcup \mathscr{U}'. \end{array}$$

Hence

$$(11) \qquad Y \times (X(\omega_1)_{X(S)})^{n+1} \backslash \bigcup \mathscr{U}' \subset \bigcup_{i=1}^{n+1} E_i$$

where

$$E_i = \left\{(y, (x_1, \dots, x_{n+1})) \in Y \times (X(\omega_1)_{X(S)})^{n+1} : x_i \in \bigcup \{Z_\gamma : \gamma \le \alpha\}\right\}.$$

The space E_i is a countable union of subsets that are Lindelöf by the inductive assumption. We conclude that $\mathscr{U}$ has a countable subcover. This completes the proof of Lemma 2.

To finish the proof of the fact that $Y \times (X_n)^n$ is Lindelöf for every hereditarily Lindelöf space Y is it sufficient to note that $(X_n)^n$ is a finite sum of elements each of which is a continuous image of a space of the form $(X(\omega_1)_{X(S)})^n$, where S is a stationary subset of ω_1 and to apply Lemma 2.

Remark 1. R. Pol pointed out to me that one can also apply similar technique to the topology defined by G. Kurepa on a Aronszajn tree (see [T]) in order to obtain another example of the kind described in this paper.

Remark 2. Using the pressing-down Lemma one can show that the product $(X_n)^{n+1}$ is not normal for all $n \in N$.

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