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ATTRACTORS OF ITERATED FUNCTION SYSTEMS

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ABSTRACT. In this paper, the question of which compact metric spaces can be attractors of hyperbolic iterated function systems on Euclidean space is studied. It is shown that given any finite-dimensional compact metric X, there is a Cantor set C such that the disjoint union of C and X is an attractor. In the process, it is proved that every such X is the Lipschitz image of a Cantor set.

1. INTRODUCTION

If X is a complete metric space and $F = \{f_1, f_2, \dots, f_k\}$ is a collection of contraction mappings of X to itself, then F is said to be a (hyperbolic) *Iterated Function System.* It is well known [4] that for such an F there is a unique compact set A such that $A = \bigcup_{i=1}^{k} f_i(A)$. A is called the *attractor* for F. In this paper, we are concerned with the question of which compact sets in Euclidean space can be realized as the attractors of some IFS. Williams [6] initially investigated the topological structure of attractors of IFS's. Hata [3] generalized Williams' work and showed that not every compactum can be realized as the attractor of an IFS, since, for example, a connected attractor must be locally connected. Hata posed the question whether every finite-dimensional locally connected continuum is the attractor of some IFS. Barnsley [1] showed to what extent every compactum can be approximated by an attractor of an IFS. As a special case, we note that it is not difficult to show that every compact subpolyhedron in $\mathbf{R}^n$ is an attractor.

We show (Theorem 4.1) that for each $n > 0$ there is a Cantor set $C^n \subset \mathbf{R}^{2n}$ with the property that if X is any compact set in $\mathbf{R}^n$, the set

$$C^n \times \{0\} \cup \{0\} \times X \subset \mathbf{R}^{3n}$$

is the attractor of an IFS. It follows that every compact, finite dimensional metric space that contains a closed and open Cantor subset can be embedded in some Euclidean space as an attractor.

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Most of our notation is standard, although we shall use $\|\cdot\|$ to denote the sum norm on $\mathbf{R}^n$. That is, if $x = (x_1, x_2, \dots, x_n)$ then $\|x\| = |x_1| + |x_2| + \cdots + |x_n|$.

2. Lipschitz mappings of Cantor sets onto compacta

A classical theorem of topology states that every compact metric space is the continuous image of the Cantor set. In this section, we construct, for each $n > 0$, a Cantor set V^n with the property that for every compactum $X \subset \mathbf{R}^n$ there is a Lipschitz mapping of V^n onto X.

To begin, fix an $n \geq 1$ and let B be the unit cube in $\mathbf{R}^n$ spanned by the standard basis vectors. Subdivide B into cubes B_i, $i \in \Gamma$, where $\Gamma = \{0, 1, \dots, 2^n - 1\}$ and $B_i = \prod_{j=0}^{n}[\varepsilon_j, 1/2 + \varepsilon_j]$ with ε_j being one half the jth binary digit of i. Subdivide the B_i's in a similar manner and continue the process so that at the kth stage, B is the union of 2^{nk} cubes $B_{i_1 i_2 \cdots i_k}$. For each $k \geq 1$, give the space of k-tuples of integers the discrete topology, let

$$V_k = \bigcup_{i_1, i_2, \dots, i_k \in \Gamma} B_{i_1 i_2 \cdots i_k} \times (i_1, i_2, \dots, i_k)$$

and define $p_k : V_{k+1} \to V_k$ by $p_k(x, i_1, i_2, \dots, i_{k+1}) = (x, i_1, i_2, \dots, i_k)$. If $\vec{i} = \{i_k\}$ is a sequence of integers with $i_k \in \Gamma$, let

$$x(\vec{i}) = \bigcap_{k=1}^{\infty} B_{i_1 i_2 \cdots i_k}.$$

In the next section, we will need the following observation, whose proof is left to the reader.

Proposition 2.1. *If $i_k = x_{k1} + x_{k2}2 + \cdots + x_{kn}2^{n-1}$, with $x_{ki} \in \{0, 1\}$, then the mth component of $x(\vec{i})$ has the base 2 expansion $.x_{1m}x_{2m}x_{3m}\cdots$.*

Let V^n be the inverse limit of the sequence

$$V_1 \xleftarrow{p_1} V_2 \xleftarrow{p_2} \cdots.$$

Note that we may view a point in V^n as a pair $(x, \vec{i})$, where $\vec{i}$ is a sequence of integers from Γ and $x = x(\vec{i})$. Define a metric ρ on V^n by

$$\rho((x, \vec{i}), (y, \vec{j})) = \| x - y \| + 1/2^k,$$

where $\|\cdot\|$ is the sum norm described in the introduction and $k + 1$ is the first index in which $\vec{i}$ and $\vec{j}$ disagree (note that if $\vec{i} = \vec{j}$ then $x = y$).

Proposition 2.2. *V^n is a Cantor set and ρ is a metric consistent with the inverse limit topology on V^n.*

Proof. Recall that if $\{X_k\}$ is a sequence of metric spaces with bounded metrics d_k, then

$$d(\{x_k\}, \{y_k\}) = \sum_{k=1}^{\infty} \frac{d_k(x_k, y_k)}{2^k}$$

is a metric on the product of the X_k's that induces the product topology. Let d_k be the metric on V_k given by

$$d_k((x, i_1, i_2, \dots, i_k), (y, j_1, j_2, \dots, j_k)) = \| x - y \| + \varepsilon;$$

where $\varepsilon = 0$ if $(i_1, i_2, \dots, i_k) = (j_1, j_2, \dots, j_k)$, 1 otherwise. Then, if $(x, \vec{i})$ and $(y, \vec{j})$ are in V^n and $k+1$ is the first index for which $\vec{i}$ and $\vec{j}$ disagree, the construction of d above gives

$$\begin{aligned} d((x, \vec{i}), (y, \vec{j})) &= \sum_{m=1}^{\infty} \frac{\| x - y \|}{2^m} + \sum_{m=k+1}^{\infty} \frac{1}{2^m} \\ &= \| x - y \| + \frac{1}{2^k} = \rho((x, \vec{i}), (y, \vec{j})). \end{aligned}$$

Thus ρ agrees with the product metric. To see that V^n is a Cantor set, note that the map $(x, \vec{i}) \to \vec{i}$ is a homeomorphism of V^n onto the space of sequences of integers from Γ. □

Now let X be a compact subset of B, and define a mapping $\phi : V^n \to X$ as follows. For each k-tuple $(i_1, i_2, \dots, i_k)$ such that $B_{i_1 i_2 \cdots i_k}$ intersects X, choose a point $x_{i_1 i_2 \cdots i_k} \in B_{i_1 i_2 \cdots i_k} \cap X$. Given $(x, \vec{i}) \in V^n$, define

$$\phi(x, \vec{i}) = \begin{cases} x_{i_1 i_2 \cdots i_k} & \text{if } x \notin X \text{ and } k \text{ is the last index with } X \cap B_{i_1 i_2 \cdots i_k} \neq \varnothing, \\ x & \text{if } x \in X. \end{cases}$$

Theorem 2.1. *ϕ is a Lipschitz map of V^n onto X.*

Proof. Given $a = (x, \vec{i})$, $b = (y, \vec{j}) \in V^n$, there are three cases to consider.

Case 1. Both x and y are in X. In this case,

$$\| \phi(a) - \phi(b) \| = \| x - y \| \leq \rho(a, b).$$

Case 2. $x \in X$, but $y \notin X$. Let $k+1$ be the first index in which $\vec{i}$ and $\vec{j}$ disagree, and let $\phi(b) = x_{j_1 j_2 \cdots j_l}$. Then $k \leq l$ and

$$\| y - x_{j_1 j_2 \cdots j_l} \| \leq n/2^l \leq n/2^k$$

since both y and $x_{j_1 j_2 \cdots j_l}$ are in $B_{j_1 j_2 \cdots j_l}$. We have

$$\begin{aligned} \| \phi(a) - \phi(b) \| = \| x - x_{j_1 j_2 \cdots j_l} \| &\leq \| x - y \| + \| y - x_{j_1 j_2 \cdots j_l} \| \\ &\leq \| x - y \| + n/2^k \leq n\rho(a, b). \end{aligned}$$

Case 3. Neither x nor y are in X. Let $\phi(a) = x_{i_1 i_2 \cdots i_l}$, $\phi(b) = x_{j_1 j_2 \cdots j_m}$, and let $k+1$ be the first index in which $\vec{i}$ and $\vec{j}$ disagree. We may assume that $l \leq m$, and we would like to have $k \leq l$. If $m \leq k$, then $\phi(a) = \phi(b)$

and there is nothing to prove. By the definition of ϕ, the case $l < k < m$ leads to a contradiction, so we assume $k \leq l$. Then

$$\begin{aligned} \| \phi(a) - \phi(b) \| &= \| x_{i_1 i_2 \cdots i_l} - x_{j_1 j_2 \cdots j_m} \| \\ &\leq \| x - y \| + \| x - x_{i_1 i_2 \cdots i_l} \| + \| y - x_{j_1 j_2 \cdots j_m} \| \\ &\leq \| x - y \| + 2n/2^k \leq 2n\rho(a, b). \end{aligned}$$

Thus ϕ is a Lipschitz map with Lipschitz constant at most $2n$. □

Since every compact set in $\mathbf{R}^n$ can be moved into B by similarity transformations, we have

Corollary 2.1. *If X is any compact set in $\mathbf{R}^n$, there is a Lipschitz map of V^n onto X.*

3. Standard Cantor sets

Let C^1 be the set of points in $\mathbf{R}^2$ of the form (x, y), where $x, y \in [0, 1]$ have base 2 expansions of the form $x = .x_1 x_2 x_3 \cdots$ and $y = .y_1 y_2 y_3 \cdots$, with $x_i = 0$ for i even and $y_i = 0$ for i odd. Define contractions h_0 and h_1 on $\mathbf{R}^2$ by

$$h_0(x, y) = \left(\frac{y}{2}, \frac{x}{2}\right) \quad \text{and} \quad h_1(x, y) = \left(\frac{y}{2} + \frac{1}{2}, \frac{x}{2}\right).$$

Proposition 3.1. *C^1 is a Cantor set and is the attractor for the IFS $F^1 = \{h_0, h_1\}$.*

Proof. It is easy to check that $h_0(C^1)$ and $h_1(C^1)$ are disjoint and that their union is C^1. Thus C^1 is the attractor and the disjointness implies that it is a Cantor set [3, Theorem 4.4]. □

If $F = \{f_1, f_2, \ldots, f_k\}$ and $G = \{g_1, g_2, \ldots, g_l\}$ are IFSs on the complete metric spaces X and Y with attractors A and B, respectively, define the IFS $F \times G$ on $X \times Y$ to be the set of all products $f_i \times g_j$.

The reader can easily verify the following proposition.

Proposition 3.2. *The attractor of $F \times G$ is $X \times Y$.*

It follows that for each $n > 1$ the n-fold product $C^n = C^1 \times C^1 \times \cdots \times C^1$ is a Cantor set in $\mathbf{R}^{2n}$ that is the attractor of the IFS $F^n = F^1 \times F^1 \times \cdots \times F^1$.

We note in passing that since F^n consists of 2^n similitudes with similarity constant $\frac{1}{2}$, the Hausdorff dimension of C^n is n [2].

In the previous section, we showed that each compactum in a finite-dimensional Euclidean space is the Lipschitz image of an abstractly defined Cantor set. The purpose of the present construction is to be able to replace V^n by the more geometrically appealing C^n. Again, we need some preliminary notation. If $u \in C^n$, we can represent u as a $2n$-tuple $u = (x^1, y^1, x^2, y^2, \ldots, x^n, y^n)$ where each x^m has the binary expansion $x^m = .x_1^m x_2^m \cdots$ and $x_k^m = 0$ if k is even and the y^m have similar expansions with $y_k^m = 0$ for k odd. Then we

can define a mapping $\psi\colon C^n \to V^n$ by $\psi(u) = (x(\vec{i}), \vec{i})$, where

$$i_k = \sum_{m=1}^{n} \frac{x_k^m + y_k^m}{2^{m-1}}.$$

Theorem 3.1. *The map $\psi\colon C^n \to V^n$ is a Lipschitz equivalence.*

Proof. It follows from the uniqueness of representation of integers and the special properties of the points in C^n that ψ is one-to-one and onto. Let u be as above, let $v = (w^1, z^1, w^2, z^2, \dots, w^n, z^n)$ be a second point, and let $\psi(u) = (a, \vec{i})$, $\psi(v) = (b, \vec{j})$. By Proposition 2.1 the mth coordinate of a is $x^m + y^m$ and the mth coordinate of b is $w^m + z^m$, so we have

$$\begin{aligned}\|a - b\| &= \sum_{m=1}^{n} |x^m + y^m - w^m - z^m| \\ &\le \sum_{m=1}^{n} |x^m - w^m| + |y^m - z^m| = \|u - v\|.\end{aligned}$$

Suppose that $k+1$ is the first index in which $\vec{i}$ and $\vec{j}$ disagree. Since a and b must have some coordinates that disagree in the $k+1$st position of their binary expansions, $\|a - b\| \ge 1/2^{k+1}$. Therefore,

$$\rho(\psi(u), \psi(v)) = \|a - b\| + \frac{1}{2^k} \le 3\|a - b\| \le 3\|u - v\|.$$

On the other hand,

$$\|u - v\| = \sum_{m=1}^{n} |x^m - w^m| + |y^m - z^m| \le \frac{2n}{2^k},$$

since x^m, w^m and y^m, z^m agree in at least their first k places. Therefore

$$\|u - v\| \le 2n\|a - b\| + \frac{2n}{2^k} = 2n\rho(\psi(u), \psi(v)). \quad \square$$

Corollary 3.1. *If X is a compact set in $\mathbf{R}^n$, there is a Lipschitz map of C^n onto X.*

4. Constructing IFSs

Recall the contractions h_0 and h_1 on $\mathbf{R}^2$ constructed in §3. For an $n > 1$ we index the members of the IFS F^n as follows. For an integer $i \in \Gamma$, let $i = \sum_{k=0}^{n-1} x_k 2^k$ be its binary representation, and let $f_i = h_{x_0} \times h_{x_1} \times \cdots \times h_{x_{n-1}}$. For $i_1, i_2, \dots, i_k \in \Gamma$, let $f_{i_1 i_2 \cdots i_k}$ be the composition $f_{i_k} f_{i_{k-1}} \cdots f_{i_1}$. It is easy to see that

Proposition 4.1. *Each* $f_{i_1 i_2 \cdots i_k}$ *is a contraction of* $\mathbf{R}^{2n}$ *with contractivity constant* $1/2^k$ *and*

$$C^n = \bigcup_{i_1, i_2, \ldots, i_k \in \Gamma} f_{i_1 i_2 \cdots i_k}(C^n)$$

for each k.

Given a compact set $X \subset \mathbf{R}^n$ with $0 \notin X$, let $C + X$ denote the union

$$C + X = (C^n \times \{0\}) \cup (\{0\} \times X) \subset \mathbf{R}^{3n}.$$

Theorem 4.1. *There is an IFS on* $\mathbf{R}^{3n}$ *whose attractor is* $C + X$.

Proof. Write $\mathbf{R}^{3n}$ as $\mathbf{R}^{2n} \times \mathbf{R}^n$ and let $\lambda : C^n \to X$ be a Lipschitz map with Lipschitz constant L. By the "Lipschitz Tietze Theorem" [5], we may assume that λ is defined on all of $\mathbf{R}^{2n}$. For each $i \in \Gamma$ let $\hat{f}_i$ be defined by

$$\hat{f}_i(x, y) = (f_i(x), 0).$$

Choose a k such that $L/2^k < 1$ and for each k-tuple $i_1, i_2, \ldots, i_k \in \Gamma$, let

$$g_{i_1 i_2 \cdots i_k}(x, y) = (0, \lambda f_{i_1 i_2 \cdots i_k}(x)).$$

Then the collection

$$\{\hat{f}_i | i \in \Gamma\} \cup \{\lambda g_{i_1 i_2 \ldots i_k} | i_1, i_2, \cdots, i_k\}$$

is an IFS whose attractor is $C + X$. □

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