MAXIMAL DOUGLAS SUBALGEBRAS
AND MINIMAL SUPPORT POINTS

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(Communicated by Paul S. Muhly)

Abstract. Let \( B \) denote a Douglas algebra. Then \( B \) has a maximal Douglas subalgebra if and only if the set of points outside the maximal ideal space of \( B \) has a minimal support point.

1. Introduction

Let \( H^\infty \) be the space of bounded analytic functions on the open unit disc \( D \), and let \( L^\infty \) be the space of bounded measurable functions on the unit circle \( \partial D \) with respect to the Lebesgue measure \( d\theta/2\pi \). Identifying \( H^\infty \) with these boundary functions, we may consider that \( H^\infty \) is an essentially sup-norm closed subalgebra of \( L^\infty \). An essentially sup-norm closed subalgebra \( B \) of \( L^\infty \) containing \( H^\infty \) is called a Douglas algebra. We denote by \( C \) the space of continuous functions on \( \partial D \). Then it is well known that \( H^\infty + C \) is a Douglas algebra and contains \( H^\infty \) as a maximal Douglas subalgebra [10]. Also in [4], Guillory, Izuchi, and Sarason showed that there is a Douglas algebra \( B \) such that \( H^\infty + C \subseteq B \subseteq L^\infty \) and \( B \) has a maximal Douglas subalgebra. On the other hand, Sundberg [11] proved that \( L^\infty \) does not have a maximal Douglas subalgebra. In this paper, we answer to a question when a given Douglas algebra has a maximal Douglas subalgebra.

For a Douglas algebra \( B \), we let \( M(B) \) denote the maximal ideal space of \( B \). A Gelfand transform of \( f \) in \( B \) is denoted by the same symbol \( f \). \( M(B) \) may be identified with a closed subset of \( M(H^\infty) \). Then \( M(L^\infty) \) becomes the Shilov boundary for \( B \) and \( M(H^\infty + C) = M(H^\infty) \setminus D \). For a point \( x \) in \( M(H^\infty) \), we denote by \( \mu_x \) its representing measure on \( M(L^\infty) \), that is, \( f(x) = \int_{M(L^\infty)} f \, d\mu_x \) for every \( f \) in \( H^\infty \). We put

\[
H^\infty_{\text{supp} \mu_x} = \{ f \in L^\infty : f|_{\text{supp} \mu_x} \in H^\infty_{\text{supp} \mu_x} \};
\]

then \( H^\infty_{\text{supp} \mu_x} \) is a Douglas algebra and

\[
M(H^\infty_{\text{supp} \mu_x}) = M(L^\infty) \cup \{ \zeta \in M(H^\infty) : \text{supp} \mu_\zeta \subseteq \text{supp} \mu_x \}.
\]
For a subset \( \Omega \) of \( M(H^\infty) \), a point \( x \) in \( \Omega \) is called a minimal support point for \( \Omega \) if there are no points \( y \) in \( \Omega \) such that \( \text{supp } \mu_y \subseteq \text{supp } \mu_x \). The following is our main theorem.

**Theorem 1.** Let \( B \) be a Douglas algebra. Then \( B \) has a maximal Douglas subalgebra if and only if \( M(H^\infty) \setminus M(B) \) has a minimal support point. If \( x \) is a minimal support point for \( M(H^\infty) \setminus M(B) \), then \( B \cap H^\infty_{\text{supp } \mu_x} \) is a maximal Douglas subalgebra of \( B \), and every maximal Douglas subalgebra is given by this form.

In §2, we prove Theorem 1. In §3, we give some applications of Theorem 1.

2. Proof of Theorem 1

A function \( \psi \) in \( H^\infty \) with \( |\psi| = 1 \) on \( M(L^\infty) \) is called inner. The famous Chang-Marshall theorem [1,9] says that if \( B \) is a Douglas algebra then \( B \) is generated by \( H^\infty \) and complex conjugates of inner functions \( \psi \in B \) and \( M(B) \) is the set of points \( x \) in \( M(H^\infty) \) such that \( |\psi(x)| = 1 \) for every inner function \( \psi \) with \( \psi \in B \), that is, \( M(B) = \{ x \in M(H^\infty) ; B \mid \text{supp } \psi = H^\infty_{\text{supp } \mu} \} \). We use these facts frequently. To prove our theorem, we need the following unpublished result of Sarason (see [3, Proposition 3.4]).

**Lemma 1.** Let \( B_1 \) and \( B_2 \) be Douglas algebras. Then \( M(B_1 \cap B_2) = M(B_1) \cup M(B_2) \).

**Proof of Theorem 1.** Let \( A \) be a maximal Douglas subalgebra of \( B \). Since \( M(B) \subseteq M(A) \), take a point \( x \) in \( M(A) \setminus M(B) \). To prove that \( x \) is a minimal support point for \( M(H^\infty) \setminus M(B) \), suppose not. Then there exists \( y \) in \( M(H^\infty) \setminus M(B) \) such that \( \text{supp } \mu_y \subseteq \text{supp } \mu_x \). Let \( C = B \cap H^\infty_{\text{supp } \mu_x} \). Since \( x \in M(A) \), it follows that \( A \cap \text{supp } \mu_x = H^\infty_{\text{supp } \mu_x} \), so that \( A \cap \text{supp } \mu_y = H^\infty_{\text{supp } \mu_y} \). Therefore we have \( A \subseteq C \subseteq B \). By Lemma 1,

\[
M(C) = M(B) \cup \{ \zeta \in M(H^\infty) ; \text{supp } \mu_\zeta \subseteq \text{supp } \mu_y \}.
\]

Since \( y \in M(C) \) and \( y \notin M(B) \), it follows that \( M(B) \subseteq M(C) \) and \( C \subseteq B \). Since \( x \notin M(C) \) and \( x \in M(A) \), it follows that \( M(C) \subseteq M(A) \) and \( A \subseteq C \). Hence \( A \subseteq C \subseteq B \), contradicting the maximality of \( A \).

Since \( A \subseteq H^\infty_{\text{supp } \mu_x} \), we have \( A \subseteq B \cap H^\infty_{\text{supp } \mu_x} \). Since \( x \notin M(B) \) and \( x \in M(B \cap H^\infty_{\text{supp } \mu_x}) \), it follows that \( B \cap H^\infty_{\text{supp } \mu_x} \subseteq A \). By the maximality of \( A \), we get \( A = B \cap H^\infty_{\text{supp } \mu_x} \).

To show the converse, let \( x \) be a minimal support point for \( M(H^\infty) \setminus M(B) \) and let \( A = B \cap H^\infty_{\text{supp } \mu_x} \). Then \( A \subseteq B \) and \( M(A) = M(B) \cup \{ \zeta \in M(H^\infty) ; \text{supp } \mu_\zeta \subseteq \text{supp } \mu_x \} \). Since \( x \in M(A) \) and \( x \notin M(B) \), we have \( A \subseteq B \). Since \( x \) is a minimal support point for \( M(H^\infty) \setminus M(B) \), if \( \text{supp } \mu_\zeta \subseteq \text{supp } \mu_x \) then \( \zeta \in M(B) \). Hence

\[
M(A) = M(B) \cup \{ \zeta \in M(H^\infty) ; \text{supp } \mu_\zeta = \text{supp } \mu_x \}.
\]

Let \( Y \) be a Douglas algebra such that \( A \subseteq Y \subseteq B \). By Chang-Marshall’s theorem, there is an inner function \( \psi \) such that \( \psi \notin A \) and \( \psi \in Y \subseteq B \). Then \( |\psi| = 1 \) on \( M(B) \) and \( |\psi(x)| < 1 \). Hence \( |\psi| < 1 \) on \( M(A) \setminus M(B) \). Since \( M(B) \subseteq M(Y) \subseteq M(A) \), it follows that \( M(B) = M(Y) \). Consequently \( B = Y \) and this implies that \( A \) is a maximal Douglas subalgebra of \( B \).
Remark. By the above proof, if $A \subseteq B$ then $A$ is a maximal Douglas subalgebra of $B$ if and only if $\text{supp } \mu_x = \text{supp } \mu_y$ for every $x, y$ in $M(A) \setminus M(B)$.

3. Applications

A Blaschke product $b$ with zeros $\{z_n\}_n$ in $D$ is called sparse if

$$\lim_{k \to \infty} \prod_{n : n \neq k} \left| \frac{z_k - z_n}{1 - z_n z_k} \right| = 1.$$  

The above condition implies that $\{z_n\}_n$ is interpolating for $H^\infty$, that is, for every bounded sequence $\{a_n\}_n$ there exists $f$ in $H^\infty$ such that $f(z_n) = a_n$ for every $n$. For every sequence $\{y_n\}_n$ in $D$ with $|y_n| \to 1$, we can find a subsequence that satisfies the sparseness condition. For a function $f$ in $H^\infty$, we put $Z(f) = \{x \in M(H^\infty + C); f(x) = 0\}$. For a subset $F$ of $L^\infty$, we denote by $[F]$ the closed subalgebra of $L^\infty$ generated by $F$.

**Theorem 2.** Let $\psi$ be an inner function such that $\psi$ is not continuous on $\partial D$. Then $[H^\infty, \psi]$ has uncountably many maximal Douglas subalgebras.

**Proof.** We have $M([H^\infty, \psi]) = \{\zeta \in M(H^\infty + C); |\psi(\zeta)| = 1\}$. For $\zeta \in M(H^\infty) \setminus M([H^\infty, \psi])$, there is a point $x$ in $Z(\psi)$ such that $\text{supp } \mu_x \subseteq \text{supp } \mu_\zeta$. Hence to study minimal support points for $M(H^\infty) \setminus M([H^\infty, \psi])$ is the same as to study minimal support points for $Z(\psi)$, so we concentrate on $Z(\psi)$.

For $x, y$ in $Z(\psi)$, we define the order as follows; $y \leq x$ if $\text{supp } \mu_y \subseteq \text{supp } \mu_x$, and $x = y$ if $\text{supp } \mu_y = \text{supp } \mu_x$. By Hoffman’s unpublished result [7], if $\text{supp } \mu_y \cap \text{supp } \mu_x \neq \emptyset$ then $\text{supp } \mu_y \subseteq \text{supp } \mu_x$ or $\text{supp } \mu_y \supset \text{supp } \mu_x$. Hence $Z(\psi)$ becomes an ordered set. Let $\{x_\alpha\}_{\alpha \in \Lambda}$ be a totally ordered subset of $Z(\psi)$. We denote by $F_\alpha$ the closure of $\{x_\beta; \beta \in \Lambda, \beta \leq \alpha\}$ in $M(H^\infty)$. Then $\{F_\alpha\}_\alpha$ is a family of compact decreasing subsets of $Z(\psi)$. Therefore, there is a point $x_0$ in $\bigcap_\alpha F_\alpha$, and we have $\text{supp } \mu_{x_0} \subseteq \text{supp } \mu_x$ and $x_0 \leq x_\alpha$. By Zorn’s lemma, $Z(\psi)$ has a minimal element; hence $M(H^\infty) \setminus M([H^\infty, \psi])$ has a minimal support point. By Theorem 1, $[H^\infty, \psi]$ has a maximal Douglas subalgebra.

Let $b$ be a sparse Blaschke product with zeros $\{z_n\}_n$ such that $|\psi(z_n)| \to 0$ $(n \to \infty)$. By [5, p. 205], $Z(b)$ is contained in the closure of $\{z_n\}_n$ in $M(H^\infty)$. Then $Z(b) \subseteq Z(\psi)$. By the second paragraph, for each $\zeta \in Z(b)$ there is a minimal element $\zeta_0$ in $Z(\psi)$ such that $\zeta_0 \leq \zeta$. By Theorem 1, $[H^\infty, \psi]$ has maximal Douglas subalgebras $\{[H^\infty, \psi] \cap H^\infty_{\text{supp } \mu_\lambda}; \zeta \in Z(b)\}$. By [4, p. 5], $\text{supp } \mu_\zeta \cap \text{supp } \mu_\lambda = \emptyset$ for distinct $\zeta, \lambda \in Z(b)$. Since $\text{supp } \mu_{\zeta_0} \subseteq \text{supp } \mu_\zeta$, it follows that $[H^\infty, \psi] \cap H^\infty_{\text{supp } \mu_\zeta} \neq [H^\infty, \psi] \cap H^\infty_{\text{supp } \mu_\lambda}$ for distinct $\zeta, \lambda \in Z(b)$. Since $Z(b)$ is homeomorphic to $\beta N \setminus N$ [5, p. 205] where $\beta N$ is the Stone-Cech compactification of integers $N$, we get our assertion.

**Theorem 3.** Let $B$ be a Douglas algebra with $B \subseteq L^\infty$. Let $\psi$ be an inner function with $\psi \notin B$. Then $[B, \psi]$ has a maximal Douglas subalgebra.

**Proof.** Let $x \in M(B) \setminus M([B, \psi])$. Then $|\psi(x)| < 1$. By the proof of Theorem 2, there is a minimal support point $y$ for $M(H^\infty) \setminus M([H^\infty, \psi])$ such that $\text{supp } \mu_y \subseteq \text{supp } \mu_x$. To prove that $y$ is a minimal support point for $M(H^\infty) \setminus M([B, \psi])$, suppose that there is a point $\zeta$ in $M(H^\infty) \setminus M([B, \psi])$ such that $\text{supp } \mu_\zeta \subseteq \text{supp } \mu_y$. Since $x \in M(B)$, we have $y \in M(B)$, so that...
\( \zeta \in M(B) \). Since \( \zeta \notin M([B, \psi]) \), \( |\psi(\zeta)| < 1 \). This contradicts the minimality of \( y \) for \( M(H^\infty) \setminus M([H^\infty, \psi]) \). Hence \( y \) is a minimal support point. By Theorem 1, we get Theorem 3.

The following theorem shows that there exists a Douglas algebra that does not have any maximal Douglas subalgebras.

**Theorem 4.** Let \( \{\psi_n\}_{n=1}^{\infty} \) be a sequence of inner functions such that \( \psi_{n+1} = 0 \) on \( \{ \zeta \in M(H^\infty + C) ; |\psi_n(\zeta)| < 1 \} \) for every \( n \). Then \( [H^\infty, \psi_n ; n = 1, 2, \ldots] \) does not have a maximal Douglas subalgebra.

**Proof.** Put \( B = [H^\infty, \psi_n ; n = 1, 2, \ldots] \). Let \( x \in M(H^\infty) \setminus M(B) \). We shall prove that there is a point \( y \) in \( M(H^\infty) \setminus M(B) \) such that \( \text{supp } \mu_y \subseteq \text{supp } \mu_x \). Since \( x \notin M(B) \), it follows that \( |\psi_n(x)| < 1 \) for some \( n \); hence \( \{ \zeta \in M(H^\infty) ; |\psi_n(\zeta)| < 1 \} \) is a nonvoid open subset of \( M(H^\infty) \). But it is not closed in \( M(H^\infty) \). By the Shilov idempotent theorem, there is a function \( g \in H^\infty \) such that \( g^2 = g \) and \( \{ \zeta \in M(H^\infty) ; g(\zeta) = 1 \} = \{ \zeta \in M(H^\infty) ; |\psi_n(\zeta)| = 1 \} \). Since \( g = 1 \) on \( M(L^\infty) \) and \( |\psi_n(x)| \neq 1 \), we have \( 0 = g(x) = \int_{M(L^\infty)} g d\mu_x = 1 \); this is impossible.

Take a point \( y \) in the closure of \( \{ \zeta \in M(H^\infty) ; |\psi_n(\zeta)| < 1 \} \) such that \( |\psi_n(y)| = 1 \). Since \( \text{supp } \mu_z \subseteq \text{supp } \mu_x \) for \( z \in M(H^\infty) \) with \( |\psi_n(z)| < 1 \), we have \( \text{supp } \mu_y \subseteq \text{supp } \mu_x \). Since \( \psi_n \) is constant on \( \text{supp } \mu_y \), and \( \psi_n \) is constant on \( \text{supp } \mu_x \), we have \( \text{supp } \mu_y \subseteq \text{supp } \mu_x \). Since \( \psi_{n+1} = 0 \) on \( \{ \zeta \in M(H^\infty + C) ; |\psi_n(\zeta)| < 1 \} \), we have \( \psi_{n+1}(y) = 0 \), so that \( y \notin M(B) \). Hence every point in \( M(H^\infty) \setminus M(B) \) is not a minimal support point for \( M(H^\infty) \setminus M(B) \). By Theorem 1, we get Theorem 4.

**Corollary 1.** There is a function \( f \) in \( L^\infty \) such that \( [H^\infty, f] \) does not have a maximal Douglas subalgebra.

**Proof.** By [11] there is a sequence of inner functions \( \{\psi_n\}_n \) that satisfy the condition of Theorem 4. Take \( f \) in \( L^\infty \) such that \( [H^\infty, f] = [H^\infty, \psi_n ; n = 1, 2, \ldots] \) (see [8, Lemma 2.2]). Then \( f \) satisfies our assertion.

For a Douglas algebra \( B \), a Douglas algebra \( A \) with \( B \subseteq A \) is called a minimal superalgebra of \( B \) if there are no Douglas algebras \( J \) such that \( B \subseteq J \subseteq A \); that is, \( B \) is a maximal Douglas subalgebra of \( A \). If \( A \) is a minimal superalgebra of \( B \), then \( A = [B, \psi] \) for some inner function \( \psi \). The following theorem is a restatement of Theorem 1 and Remark.

**Theorem 5.** Let \( B \) be a Douglas algebra. Then \( B \) has a minimal superalgebra if and only if there is an inner function \( \psi \), \( \psi \notin B \) such that \( \text{supp } \mu_x = \text{supp } \mu_y \) for every \( x, y \) in \( \{ \zeta \in M(B) ; |\psi(\zeta)| < 1 \} \). Under the above situation, \( [B, \psi] \) becomes a minimal superalgebra of \( B \).

**Corollary 2.** For every function \( f \) in \( L^\infty \) with \( f \notin H^\infty \), \( [H^\infty, f] \) does not have any minimal superalgebras.

**Proof.** Put \( B = [H^\infty, f] \). By [8, Lemma 2.2], there is a sequence of inner functions \( \{\psi_n\}_n \) such that \( B = [H^\infty, \psi_n ; n = 1, 2, \ldots] \). Then \( M(B) \) is a closed \( G_\delta \)-subset of \( M(H^\infty) \). Let \( g \) be a nonnegative continuous function on \( M(H^\infty) \) such that \( g = 1 \) on \( M(B) \) and \( 0 \leq g < 1 \) on \( M(H^\infty) \setminus M(B) \). To prove our assertion, let \( \psi \) be an inner function such that \( \psi \notin B \). Then
\[ Z(\psi) \cap M(B) \neq \emptyset. \] By the corona theorem (see [2, p. 318]), there is a sequence \( \{z_n\}_n \) in \( D \) such that \( g(z_n) \to 1 \) and \( \psi(z_n) \to 0 \). Let \( b \) be a Blaschke product with zeros \( \{z_n\}_n \). We may assume that \( b \) is sparse. Then we have \( Z(b) \subset M(B) \cap Z(\psi) \) and \( \text{supp}\ \mu_x \neq \text{supp}\ \mu_y \) for \( x, y \) in \( Z(b) \), \( x \neq y \). By Theorem 5, we get our assertion.

**ACKNOWLEDGMENT**

The authors would like to thank the referee for shortening the proof of Theorem 1.

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