

## ON ARCHIMEDEAN ORDERED VECTOR SPACES AND A CHARACTERIZATION OF SIMPLICES

GERHARD GIERZ AND BORIS SHEKHTMAN

(Communicated by William J. Davis)

**ABSTRACT.** We show that a convex subset  $K$  of a linear space is a simplex if and only if it is line compact and every nonempty intersection of two translates of  $K$  is a homothet of  $K$ . This answers a problem posed by Rosenthal. The proof uses a reformulation of this problem in terms of Archimedean ordered spaces

### INTRODUCTION

Let  $K$  be a convex subset of a linear space  $E$ . If  $K \times \{1\}$  is the base for a lattice cone in  $X \times \mathfrak{R}$ , then  $K$  is called a simplex (see [7] for the definitions). A remarkable result of Kendall [4] shows that  $K$  is a simplex if and only if  $K$  is line-compact<sup>1</sup> and the nonempty intersection of two homothets<sup>2</sup> of  $K$  is a homothet of  $K$ . Moreover, every simplex has the property that it is line compact and every nonempty intersection of two translates of  $K$  is a homothet of  $K$ . It is an open problem whether this last condition fully characterizes simplices, but a result of Rosenthal [7] shows that this is the case at least for  $\sigma$ -convex subsets of topological vector spaces. Using Archimedean ordered spaces and their known relation to simplices, Rosenthal [7] reformulated this open problem as follows.

Let  $E$  be an ordered vector space. Given a positive function  $\mu: E \rightarrow \mathfrak{R}$ , we say that  $E$  has a  $\mu$ -lattice structure if  $\mu(u_1) = \mu(u_2)$  implies that  $u_1 \vee u_2$  exists in  $E$ .

Note that in an ordered vector space the existence of  $-u_1 \vee H - u_2$  implies that  $u_1 \wedge u_2$  exists and is equal to  $-(-u_1 \vee -u_2)$ . Hence in a  $\mu$ -lattice the infimum  $u_1 \wedge u_2$  exists whenever  $\mu(u_1) = \mu(u_2)$ .

Recall that a positive functional  $\mu$  is called strictly positive if  $u \geq 0$  and  $\mu(u) = 0$  implies that  $u = 0$ .

Rosenthal has shown that an affirmative answer to the following question leads to a positive solution of the open problem concerning simplices as stated above.

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Received by the editors December 4, 1990 and, in revised form, February 21, 1991.

1991 *Mathematics Subject Classification*. Primary 46A55, 46A40.

*Key words and phrases*. Simplices, Archimedean ordered spaces, sublattices of  $C(K)$ .

<sup>1</sup>  $K$  is line-compact if the intersection of every line with  $K$  is compact.

<sup>2</sup> A homothet of  $K$  is a set of the form  $a + r \cdot K$ , where  $r \geq 0$  is a positive constant.

**Problem.** Let  $E$  be an Archimedean ordered space. If  $E$  has a  $\mu$ -lattice structure for a strictly positive linear functional  $\mu$  on  $E$ , is it necessarily a vector lattice?

In this note we give an affirmative answer to this problem (Theorem 3.5).

For definitions and results concerning vector lattices and Banach lattices, see [5, 3].

## 2. $\mu$ -SUBLATTICES OF $C(K)$

In the following, let  $E \subseteq C([0, 1])$  be a subspace such that (i)  $1, x \in E$  and (ii) there is a positive measure  $\mu$  on  $[0, 1]$  such that  $0, 1 \in \text{supp}(\mu)$  and such that for all  $f, g \in E$  with  $\mu(f) = \mu(g)$  we have  $\text{sup}(f, g) \in E$ . We would like to show that  $E$  contains all piecewise linear functions.

We need some notation. For a point  $a \in [0, 1]$ , define maps  $\lambda_a, \rho_a \in C([0, 1])$  by

$$\lambda_a(x) = \begin{cases} a - x & \text{if } x \leq a, \\ 0 & \text{else} \end{cases}$$

and

$$\rho_a(x) = \begin{cases} x - a & \text{if } x \geq a, \\ 0 & \text{else.} \end{cases}$$

Since  $\lambda_a(x) - \rho_a(x) = a - x = a \cdot 1 - x$ , we have  $\lambda_a \in E$  if and only if  $\rho_a \in E$ . Also, since  $\lambda_a(x) = \text{sup}\{(a \cdot 1 - x), 0\}$ , condition (ii) implies that there is a number  $a_0$  with  $0 < a_0 < 1$  such that  $\lambda_{a_0}, \rho_{a_0} \in E$ .

**2.1. Proposition.** *Let  $E \subseteq C([0, 1])$  be a subspace such that*

- (i)  $1, x \in E$ ;
- (ii) *there is a positive measure  $\mu$  on  $[0, 1]$  such that  $0, 1 \in \text{supp}(\mu)$  and such that  $E$  is a  $\mu$ -lattice.*

*Then  $E$  contains all piecewise linear functions.*

*Proof.* Every piecewise linear function is a linear combination of functions of the forms  $\lambda_b$  and  $\rho_b$ ,  $0 < b < 1$ . Hence, we have to show that  $\lambda_b, \rho_b \in E$  for all  $b$  with  $0 < b < 1$ . Let  $a_0$  be a number with  $0 < a_0 < 1$  such that  $\lambda_{a_0}, \rho_{a_0} \in E$ , and let  $0 < b < 1$  be given. We shall assume that  $a_0 < b$ ; the case where  $b < a_0$  is treated similarly.

Consider the function  $f$  given by  $f(x) = x - b$  and let  $r = \int f(x) d\mu$ . Then, since  $\mu$  is positive and  $0 \in \text{supp}(\mu)$ , it follows that  $\int \lambda_{a_0}(x) d\mu > 0$ , and we can find a number  $s$  such that

$$r = s \int \lambda_{a_0}(x) d\mu.$$

From our assumptions on  $E$  it follows that

$$g_1 = \text{sup}(f, s \cdot \lambda_{a_0}) - s \cdot \lambda_{a_0} \in E.$$

We will consider two cases:

- (i)  $-b/a_0 \leq s$ . In this case,  $\rho_b = g_1 \in E$ .
- (ii)  $s < -b/a_0$ . In this case,  $g_1 = r_1 \cdot \lambda_{a_1} + \rho_b$  where

$$r_1 = -(1 + s) \quad \text{and} \quad a_1 = \frac{b + sa_0}{1 + s} \leq a_0.$$

Let

$$R = \int \rho_b d\mu, \quad S = \int \lambda_{a_0} d\mu.$$

Since  $\rho_b \leq g_1$ , we have  $\int g_1 d\mu \geq R$ . Again, we can find a number  $s_1$  so that

$$s_1 \int \lambda_{a_0} d\mu = \int g_1 d\mu,$$

and it follows that  $s_1 \geq R/S$ . It follows that  $a_0 \cdot R/S \leq s_1 \cdot a_0 = s_1 \cdot \lambda_{a_0}(0)$ . Define

$$g_2 = \sup(g_1, s_1 \cdot \lambda_{a_0}) - s_1 \cdot \lambda_{a_0}.$$

Then  $g_2 \in E$ , and either  $g_2 = \rho_b \in E$  (in the case where  $s_1 \cdot a_0 \geq r_1 \cdot a_1$ ) or  $g_2 = r_2 \cdot \lambda_{a_2} + \rho_b$  where  $a_2 \leq a_1 \leq a_0$  and

$$r_2 \cdot a_2 = r_1 \cdot a_1 - s_1 \cdot a_0 \leq r_1 \cdot a_1 - (R/S) \cdot a_0.$$

We continue in this way with  $r_2$  and  $a_2$  in the place of  $r_1$  and  $a_1$  until we finally find an index  $n$  such that  $s_n \cdot a_0 \geq r_n \cdot a_n$ , and therefore  $g_{n+1} = \rho_b \in E$ . Note that this procedure has to terminate after finitely many steps since

$$r_n \cdot a_n \leq r_{n-1} \cdot a_{n-1} - (R/S) \cdot a_0 \quad \text{and} \quad s_n \geq R/S. \quad \square$$

In the following result, we generalize the domain of the functions slightly. Before we do this, let us make some remarks concerning positive measures. Let  $K$  and  $K'$  be compact Hausdorff spaces, and let  $\phi: K \rightarrow K'$  be a continuous map. If  $\nu$  is a measure on  $K$ , then define a measure  $\nu'$  on  $K'$  by

$$\int d\nu' = \int (f \circ \phi) d\nu.$$

Then  $\nu' = T_\phi^*(\nu)$ , where  $T_\phi^*$  is the adjoint to the operator

$$T_\phi: C(K') \rightarrow C(K), \\ f \mapsto f \circ \phi.$$

The supports of  $\nu$  and  $\nu'$  are related by the equation

$$\text{supp}(\nu') = \phi(\text{supp}(\nu)).$$

**2.2. Proposition.** *Let  $E \subseteq C([a, b])$  be a subspace such that*

- (i)  $1, x \in E$ ;
- (ii) *there is a positive measure  $\mu$  on  $[0, 1]$  such that  $a, b \in \text{supp}(\mu)$  and such that  $E$  is a  $\mu$ -lattice.*

*Then  $E$  contains all piecewise linear functions.*

*Proof.* Define a map

$$\phi: [0, 1] \rightarrow [a, b], \quad x \mapsto a + (b - a) \cdot x.$$

Then the map

$$T_\phi: C([a, b]) \rightarrow C([0, 1]), \quad f \mapsto f \circ \phi$$

is a (norm-preserving) linear bijection that respects the lattice structure of  $C([a, b])$  and that also sends the space of all piecewise linear functions on  $C([a, b])$  onto the space of all piecewise linear functions on  $C([0, 1])$ . Moreover,  $\mu$  is a positive measure on  $C([0, 1])$  if and only if it is of the form

$\mu = T_\phi^*(\nu)$ , where  $\nu$  is a positive measure on  $[a, b]$ . Furthermore, since  $\text{supp}(\mu) = \phi(\text{supp}(\nu))$ , we have  $0, 1 \in \text{supp}(\mu)$  if and only if  $a, b \in \text{supp}(\nu)$ . Hence Proposition 2.2 follows from 2.1.  $\square$

**2.3. Theorem.** *Let  $K$  be a compact Hausdorff space, let  $E \subseteq C(K)$  be a linear subspace containing a strictly positive function  $e$ , and let  $\mu$  be a strictly positive measure on  $K$  such that  $E$  is a  $\mu$ -lattice. Then  $E$  is a sublattice of  $C(K)$ .*

*Proof.* First, consider the map

$$T_{e^{-1}}: C(K) \rightarrow C(K), \quad f \mapsto e^{-1} \cdot f.$$

This map is linear, bijective, and preserves the lattice structure. Hence, we may replace  $E$  by  $T_{e^{-1}}(E)$ ,  $e$  by 1, and  $\mu$  by  $e \cdot \mu$  in the statement of the theorem. It follows that we may assume w.l.o.g. that  $1 \in E$ .

Now let  $\phi \in E$  be arbitrary. We have to show that  $|\phi| \in E$ . Let

$$a = \min_{0 \leq x \leq 1} \phi(x), \quad b = \max_{0 \leq x \leq 1} \phi(x).$$

Consider the linear operator

$$T_\phi: C([a, b]) \rightarrow C(K) \\ f \mapsto f \circ \phi.$$

Again, this operator is linear and a lattice homomorphism (in the sense that  $T_\phi(|f|) = |T_\phi(f)|$  for all continuous  $f \in C([a, b])$ ). Let  $F = T_\phi^{-1}(E)$  and let  $\mu' = T_\phi^*(\mu)$ . Then  $F$  is a linear subspace of  $C([a, b])$  with  $1 \in F$  (since  $T_\phi(1) = 1 \circ \phi = 1$ ), and we have  $x \in F$ ; indeed,  $x \in F$  means that  $i \in F$ , where  $i(x) = x$ . But  $i \in F$  is equivalent to  $T_\phi(i) = i \circ \phi = \phi \in E$ , and the last statement is true by our assumptions. Moreover,  $\mu'$  is positive; if  $f \in C[a, b]$  is positive, then  $f \circ \phi$  is positive, and we obtain  $\mu'(f) = \mu(f \circ \phi) \geq 0$ . Since  $\mu$  is strictly positive,  $\text{supp}(\mu) = K$ , hence  $\text{supp}(\mu') = \phi(K)$ . It follows that  $a, b \in \text{supp}(\mu')$ . Lastly, assume that  $f, g \in F$  and that  $\mu'(f) = \mu'(g)$ . Then  $\mu(T_\phi(f)) = \mu(T_\phi(g))$  and  $T_\phi(f), T_\phi(g) \in E$ . Hence, by our assumptions on  $E$ , we conclude that  $T_\phi(\sup(f, g)) = \sup(T_\phi(f), T_\phi(g)) \in E$ , and therefore  $\sup(f, g) \in F$ . It follows from Proposition 2.2 that  $F$  contains all piecewise linear functions. In particular,  $|i| \in F$  (where  $|i|(x) = |x|$  for all  $a \leq x \leq b$ ). By the definition of  $F$ , we conclude that  $|\phi| = |T_\phi(i)| = T_\phi(|i|) \in E$ .  $\square$

**2.4. Proposition.** *Let  $E \subseteq C(K)$  be a subspace such that  $1 \in E$ . Assume that  $E$  separates the points of  $K$ . Further, let  $\phi: E \rightarrow \mathfrak{R}$  be a strictly positive functional on  $E$  such that  $E$  is a  $\phi$ -lattice. Then  $\phi$  can be extended to a strictly positive linear functional  $\phi': C(K) \rightarrow \mathfrak{R}$ .*

*Proof.* Firstly, we may assume that  $\phi(1) = 1$ . By the vector lattice version of the Hahn-Banach Theorem (see Proposition II.4.4 of [3] with  $U = \{f \in C(K): -1 \leq f \leq 1\}$ ), we can extend  $\phi$  to a positive functional  $\phi': C(K) \rightarrow \mathfrak{R}$ , and it suffices to show that each such extension is strictly positive. Actually, there is only one such extension, since it will follow later that  $E$  is dense in  $C(K)$ , and the density of  $E$  in  $C(K)$  implies that this extension has to be strictly positive, but we cannot use the density of  $E$  at this point.

Since every positive functional is bounded and hence given by a measure on  $K$ , we find that  $\phi'(f) = \int f d\mu$  for some positive measure  $\mu$  on  $K$ . In order

to show that  $\phi'$  is strictly positive, we have to show that  $\text{supp}(\mu) = K$ . Assume not, and let  $x_0 \in K \setminus \text{supp}(K)$ . We will construct a positive element  $f_0 \in E$  such that  $f_0(x_0) > 0$  and  $f_0(y) = 0$  for all  $y \in \text{supp}(\mu)$ . This will lead to the contradiction  $0 < \phi(f_0) = \phi'(f) = \int f_0 d\mu = \int_{\text{supp}(\mu)} f_0 d\mu = 0$ .

As a first step, we show that for every  $x \in K$  there is an open neighborhood  $U$  of  $x$  and a positive element  $f \in E$  such that  $0 \neq f$  and  $f(y) = 0$  for all  $y \in U$ . In order to construct  $f$ , pick any nonconstant function  $g \in E$  and consider the element  $g - \phi(g) \cdot 1$ . There are two cases to consider.

1.  $g(x) \neq \phi(g)$ . If  $g(x) < \phi(g)$ , then since  $\phi(g - \phi(g) \cdot 1) = 0$ , the element  $f = (g - \phi(g) \cdot 1)_+ = (g - \phi(g) \cdot 1) \vee 0$  belongs to  $E$  and is 0 on a neighborhood of  $U$ . Moreover, since  $g$  is not constant,  $g - \phi(g) \cdot 1 \neq 0$ . Since  $g - \phi(g) \cdot 1 < 0$  would imply that  $\phi(g - \phi(g) \cdot 1) < 0$ , we conclude that  $g - \phi(g) \cdot 1$  is not negative, hence there is at least one point  $x_1$  for which  $f(x_1) = (g - \phi(g) \cdot 1)(x_1) > 0$ . We conclude that  $0 < f$ . If  $g(x) > \phi(g)$ , then we replace  $g$  by  $-g$ .

2.  $g(x) = \phi(g)$ . In this case, the same construction as in case (1) delivers a function  $f' > 0$  such that  $f'(x) = 0$ . We now replace  $g$  by  $f'$ , and for this new  $g$  we have  $g(x) = 0 < \phi(g)$ . Hence case (1) applies to the new function  $g$ , and we also find a function  $f > 0$  such that  $f(y) = 0$  for all  $y$  in a neighborhood of  $x$ .

In the next step, we show that for every  $x \in \text{supp} \mu$  there is a function  $f_x \in E$  such that  $0 \leq f_x$ ,  $f_x(x_0) > 0$  and such that  $f_x$  vanishes on a neighborhood  $U_x$  of  $x$ . In order to construct  $f_x$ , pick any function  $g$  such that  $g(x_0) \neq g(x)$ . Such a function exists since  $E$  separates the points of  $K$ . After subtracting a multiple of the constant function 1 and multiplying by  $-1$ , if necessary, we may assume that  $g(x_0) > 0 > g(x)$ . Let  $r = \phi(g)$ . If  $r \geq 0$ , then pick any positive function  $0 \neq h \in E$  with  $h(x_0) = 0$  and  $\phi(h) = 1$ . If  $r \leq 0$ , then pick a negative function  $0 \neq h \in E$  that vanishes on a neighborhood of  $x$  and satisfies  $\phi(h) = -1$ . In both cases, the function  $h$  exists by the previous step. Now let  $g' = g - |r| \cdot h$ . Then  $\phi(g') = 0$ ,  $g'(x_0) > 0$ , and  $g'$  is negative on a neighborhood of  $x$ . Hence, by our assumptions on  $E$ ,  $f_x = g' \vee 0 \in E$ ,  $f_x(x_0) > 0$ , and  $f_x$  vanishes on a neighborhood  $U_x$  of  $x$ .

Now we continue with a standard compactness argument. Finitely many of the neighborhoods  $U_x$  cover the compact set  $\text{supp} \mu$ , say  $\text{supp} \mu \subseteq U_{x_1} \cup \dots \cup U_{x_n}$ . Clearly, there are strictly positive constants  $r_1$  and  $r_2$  such that  $\mu(r_1 f_{x_1}) = \mu(r_2 f_{x_2})$ , which implies that  $r_1 f_{x_1} \wedge r_2 f_{x_2} \in E$ . Continuing this process inductively yields constants  $r_1, \dots, r_n > 0$  such that  $f_0 = r_1 f_{x_1} \wedge \dots \wedge r_n f_{x_n} \in E$ . This function  $f_0$  vanishes on  $\text{supp} \mu$  and is strictly positive at  $x_0$ .  $\square$

**2.5. Corollary.** *Let  $K$  be a compact Hausdorff space, and let  $E \subseteq C(K)$  be a linear subspace that separates the points of  $K$ . Assume that  $1 \in E$ . If there is a strictly positive functional  $\phi: E \rightarrow \mathbb{R}$  such that  $E$  is a  $\phi$ -lattice, then  $E$  is a sublattice of  $C(K)$ .*

### 3. ARCHIMEDEAN ORDERED VECTOR SPACES

In this section, we will show that every Archimedean ordered vector space that is also a  $\mu$ -lattice for a certain strictly positive functional  $\mu$  is actually a vector lattice.

We will start our discussion with order unit spaces. This special case can be reduced to 2.5. The arguments are to a large extent standard in the theory of compact convex sets and boundary integrals, see also [1]. If  $(E, e)$  is an Archimedean ordered vector space with an order unit  $e$ , let  $S(E)$  denote the state space of  $E$ , that is  $S = \{\phi: E \rightarrow \mathbb{R} \mid \phi \text{ is positive, and } \phi(e) = 1\}$ . When equipped with the weak- $*$ -topology,  $S$  is a compact convex set. The following result is a consequence of Herve's Theorem (see also Proposition 1.4.1 in [1]); the proof is a variation of the proof of Theorem II.1.9 in [1].

**3.1. Theorem.** *Let  $(E, e)$  be an Archimedean ordered vector space with order unit  $e$ , and let  $p \in \partial S(E)$  be an extreme point of  $S(E)$ . If  $a, b \in E$  and  $a \vee b$  exists, then  $p(a \vee b) = \max\{p(a), p(b)\}$ . Moreover, if  $\phi$  belongs to the weak- $*$ -closure of  $\partial S(E)$ , then  $\phi(e) = 1$  and  $\phi$  preserves all existing suprema.*

*Proof.* The proof of the first half of the theorem is an exact copy of the corresponding part of Theorem II.1.9 in [1]; one only has to remark that it is enough to postulate the existence of  $a \vee b$  and that the full lattice structure is never used. For the second half, let  $(p_i)_{i \in I}$  be a net of extreme points of  $S(E)$  that converges to  $\phi$  in the weak- $*$ -topology. Then  $\phi(e) = \lim p_i(e) = \lim 1 = 1$  and  $\phi(a \vee b) = \lim p_i(a \vee b) = \lim \max\{p_i(a), p_i(b)\} = \max\{\lim p_i(a), \lim p_i(b)\} = \max\{\phi(a), \phi(b)\}$  whenever  $a \vee b$  exists.  $\square$

**3.2. Theorem.** *Let  $(E, e)$  be an Archimedean ordered vector space with order unit  $e$ . Let  $K$  be the weak- $*$ -closure of  $\partial S(E)$ . Then  $(E, e)$  is order isomorphic to a subspace  $F$  of  $C(K)$  under an isomorphism  $\Psi: E \rightarrow F$  such that*

- (i)  $\Psi(e) = 1$ .
- (ii) *If  $a \vee b$  exists in  $E$ , then the pointwise supremum of  $\Psi(u)$  and  $\Psi(v)$  belongs to  $F$  and  $\Psi(a \vee b) = \Psi(a) \vee \Psi(b)$ .*
- (iii)  $F$  separates the points of  $K$ .

*In addition, if  $E$  admits a strictly positive functional  $\mu$  such that  $E$  is a  $\mu$ -lattice, then  $F$  is a  $\mu \circ \Psi^{-1}$ -lattice.*

*Proof.* The proof of this result is an easy application of Kadison's Theorem (see also II.1.8 in [1]) in connection with 3.1.  $\square$

We now can show the following

**3.3. Corollary.** *Let  $(E, e)$  be an Archimedean ordered vector space with order unit  $e$ . If there is a strictly positive measure  $\mu$  on  $E$  such that  $E$  is a  $\mu$ -lattice, then  $E$  is a vector lattice.*

*Proof.* This statement follows from 2.5 and 3.2 and the observation that an ordered vector space that is order isomorphic to a vector lattice is a vector lattice in its own right.  $\square$

Before we prove our main result, we need one additional lemma.

**3.4. Lemma.** *Let  $E$  be an ordered vector space, and assume that the positive cone  $C = \{x \in E: 0 \leq x\}$  contains at least one nonzero element. If there is a strictly positive linear functional  $\mu: E \rightarrow \mathbb{R}$  such that  $E$  is a  $\mu$ -lattice, then the positive cone is generating in the sense that  $E = C - C$ .*

*Proof.* Let  $0 \neq u \in C$ , and let  $x \in E$  be arbitrary. If  $\mu(x) \geq 0$ , then  $0, x \leq x \vee (\mu(x)/\mu(u)) \cdot u \in E$ . It follows that

$$x = (\mu(x)/\mu(u)) \cdot u - ((\mu(x)/\mu(u)) \cdot u - x) \in C - C.$$

If  $\mu(x) \leq 0$ , then  $\mu(-x) \geq 0$ , hence  $-x \in C - C$  by the previous argument, and thus  $x \in C - C$ , since  $C - C$  is a linear subspace.  $\square$

We are now able to show

**3.5. Theorem.** *Let  $E$  be an Archimedean ordered vector space, and assume that the positive cone of  $E$  is generating. If there exists a strictly positive functional  $\mu$  on  $E$  such that  $E$  is a  $\mu$ -lattice, then  $E$  is a vector lattice.*

*Proof.* For each positive  $u \in E$  let  $E_u = \bigcup_{n>0} \{x \in E : nu \leq x \leq nu\}$  be the order ideal generated by  $u$ . Then  $(E_u, u)$  is an Archimedean ordered vector space with order unit  $u$ . Moreover, the restriction of  $\mu$  to  $E_u$  defines a strictly positive functional on  $E_u$ , and  $E_u$  is a  $\mu$ -lattice with respect to this functional. Hence each  $E_u$  is a vector lattice by 3.3. Since  $E = \bigcup_{u>0} E_u$ , it can be expected that  $E$  is also a vector lattice; we just have to show that the supremum of the element  $x \in E$  with 0 does not depend on the order ideal  $E_u$  in which it is computed. Let us denote the supremum of  $x$  and 0 in  $E_u$  by  $x_u$ . Let  $u$  and  $v$  be two positive elements such that  $x \in E_u \cap E_v$ . If  $u \leq v$ , then  $E_u \subseteq E_v$ , and hence it follows that  $x_v \leq x_u$  since  $x_u \in E_v$  is an upper bound of  $x$  and 0. Because  $E_u$  is an order ideal, it follows that  $x_v \in E_u$ , and since  $x_u$  is the least upper bound of  $x$  and 0 in  $E_u$ , we conclude that  $x_u \leq x_v$ , i.e.,  $x_u = x_v$ . If  $u$  and  $v$  are arbitrary positive elements such that  $x \in E_u \cap E_v$ , then also  $x \in E_{u+v}$ , and we have just argued that  $x_u = x_{u+v} = x_v$ . Therefore the supremum of  $x$  and 0 exists in  $E$ , and  $E$  is a vector lattice.  $\square$

In connection with Rosenthal's result in [7] we now obtain

**3.6. Corollary.** *Let  $K$  be a line-compact convex set such that the nonempty intersection of two translates of  $K$  is a homothet of  $K$ . Then  $K$  is a simplex.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CALIFORNIA 92521  
E-mail address: gierz@ucrmath.ucr.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620  
E-mail address: boris@gauss.math.usf.edu