PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 2, October 1992

FREE SUBSEMIGROUPS OF DOMAINS

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(Communicated by Maurice Auslander)

ABSTRACT. It is proved that the multiplicative semigroup of the ring of polynomials in two commuting indeterminates over a noncommutative domain contains a noncommutative free subsemigroup.

The question whether the multiplicative group of a division ring D contains a free noncommutative subgroup has been raised by Lichtman [4]. The answer is positive when D is finite-dimensional over its center Z. A more general result is given in [2]. The question is studied further in [3], but the general case is still open. Makar-Limanov raised a seemingly simpler question: Does $D^* = D - \{0\}$ contain a free noncommutative subsemigroup? In [5] he proved that the answer is positive when Z is uncountable.

In the present note we propose a more general question: Given a noncommutative domain D (with 1) does D^* contain a free noncommutative subsemigroup? Using a recent result [1, Theorem 3] it is possible to extend Makar-Limanov's result to noncommutative domains. We prove a stronger result that has the advantage that it does not make any assumptions on the center. It says that if R is a noncommutative domain and u, v are commuting indeterminates over R, then the multiplicative semigroup of the ring of polynomials $R[u, v]$ contains a free noncommutative subsemigroup. The case of a domain with uncountable center then becomes an easy corollary.

Let W be the free semigroup of words in two letters x, y. A nonempty word can be written in the form $x^{i_1}y^{j_1}\cdots x^{i_r}y^{j_r}$ with $r, j_1, \dots, i_r \geq 1$ and $i_1, j_r \geq 0$. A relation between two elements a, b of a semigroup is a pair (w_1, w_2) of two distinct words $w_1, w_2 \in W$. Since we are interested in the semigroup S^* where S is a domain, we may consider only relations (w_1, w_2) such that w_1, w_2 do not end (and do not start) with the same letter. A relation is said to be homogeneous if w_1 and w_2 have the same degree in x and the same degree in y.

In what follows, R will be a domain and $R[u, v]$ the polynomial ring in two commuting indeterminates u, v over R. The subring of R generated by 1 is denoted by P, and $P\langle x, y\rangle$ is the free algebra in x, y over P.

Received by the editors October 12, 1990 and, in revised form, March 25, 1991.
1991 *Mathematics Subject Classification.* Primary 16U10; Secondary 20M05.

Lemma. *Given a, $b \in R$, if (w_1, w_2) is a relation between $a+u$ and $b+v(a+u)$, considered as elements of $R[u, v]$, then this relation is homogeneous and $(ad_a)^n b = 0$ where $n+1$ is the total degree of the relation.*

The proof we give is similar to that of Lemma 2 in [5]. There is some flow in that proof. The assumption made in the statement (the relation holds for $x+c$, $y+dx$) is not the one used in the proof (the relation holds for $x+c$, $y+d(x+c)$). For this reason, we repeat some of the arguments and we even get a simpler proof due to our reduction to relations with distinct last letters.

Proof. The equality $w_1(a+u, b+v(a+u)) = w_2(a+u, b+v(a+u))$ yields that the two sides have the same v-degree and the same u-degree, and this implies that (w_1, w_2) is homogeneous. We may assume by symmetry that w_1 ends with y and w_2 ends with x. If the y-degree of the relation is j then we have

$$w_1(x, y+vx) - w_2(x, y+vx) = \sum_{q=1}^{j} f_q(x, y) v^{j-q} \tag{1}$$

where $f_q(x, y)$ belongs to $P\langle x, y\rangle$ and is homogeneous in x and y, and of degree q in y. Moreover $f_q(a+u, b) = 0$ and, in particular, $f_1(a+u, b) = 0$. We have

$$f_1(x, y) = \sum_{m=0}^{n} \alpha_m x^m y x^{n-m} \tag{2}$$

where $n+1$ is the total degree of the relation and $\alpha_n = 1$ since w_1 ends with y.

Now we proceed to prove by induction on n that if for two elements a, $b \in R$ there exists $f_1(x, y) \in P\langle x, y\rangle$ as in (2) with $\alpha_n = 1$ such that $f_1(a+u, b) = 0$, then $(ad_a)^n b = 0$. We may assume that $b \neq 0$.

If $n = 1$ then $\alpha_0 b(a+u) + (a+u)b = 0$ so $(\alpha_0 + 1)bu = 0$ and $\alpha_0 = -1$ since $b \neq 0$. Thus we have $(ad_a)b = ab - ba = 0$. To proceed from $n-1$ to n, we show that there exists $g(x, y) = \sum_{m=0}^{n-1} \beta_m x^m y x^{n-1-m} \in P\langle x, y\rangle$ with $\beta_{n-1} = 1$ such that $g(a+u, b_1) = 0$ where $b_1 = (ad_a)b$. By induction, this will imply $(ad_a)^{n-1} b_1 = 0$ and therefore $(ad_a)^n b = 0$.

To get $g(x, y)$, let $y_1 = [x, y]$ so $xy = yx + y_1$. Using this relation we transform the left-hand side of (2) into

$$\sum_{m=0}^{n-1} \beta_m x^m y_1 x^{n-1-m} + \beta y x^n . \tag{3}$$

Since $\alpha_n = 1$, we get $\beta_{n-1} = 1$. Let $g(x, y) = \sum_{m=0}^{n-1} \beta_m x^m y x^{n-1-m}$, then, since $f_1(a+u, b) = 0$, we have $g(a+u, b_1) + \beta b(a+u)^n = 0$ where $b_1 = [a, b]$. But the u-degree of $g(a+u, b_1)$ is at most $n-1$, so $\beta = 0$ and $g(a+u, b_1) = 0$. This completes the proof of the lemma.

Theorem. *Let R be a noncommutative domain and $R[u, v]$ the ring of polynomials in two commuting indeterminates over R. Then the multiplicative semigroup of $R[u, v]$ contains a free noncommutative subsemigroup.*

Proof. If the result is false, then given a, $b \in R$ the elements $a+u$, $b+v(a+u)$ of $R[u, v]$ satisfy a relation. It follows by the lemma that $(ad_a)^n b = 0$.

But by [1, Theorem 3] a domain R with the last property is commutative, a contradiction.

Corollary. *Let S be a noncommutative domain with uncountable center Z. Then S^* contains a free noncommutative subsemigroup.*

Proof. Applying the theorem, it suffices to prove that S contains a subring isomorphic to a ring of polynomials $R[u, v]$ where R is a noncommutative subring of S.

Let $a, b \in S$ be such that $ab \neq ba$, and let R be the subring generated by a, b, and 1. Thus R is a countable domain. Since a nonzero polynomial of degree n in $R[u]$ has at most n roots in Z, we get that the set of roots belonging to Z, of all the nonzero polynomials in $R[u]$ is countable. Since Z is uncountable, there exists an element $u_1 \in Z$ that is transcendental over R, so $R[u_1] \simeq R[u]$. Starting with the domain $R_1 = R[u_1]$, which is also countable, we get by the same argument, that there exists an element $v_1 \in Z$ transcendental over R_1. So $R_1[v_1] \simeq R[u][v]$ and, therefore, $R[u_1, v_1] \simeq R[u, v]$. Thus $R[u_1, v_1]$ is the required subring of S.

We conclude with several remarks.

1. We have not assumed that the center is a field, but our question may be reduced to this case. Indeed, by localizing at the center we get a domain with the center a field and if the answer is positive for the localized domain then it is positive for the given domain. This follows since one can show that if two elements in D satisfy a relation then they also satisfy a homogeneous relation.

2. Our domain may be assumed to be right and left *Ore* for if this is not the case, it contains not only a noncommutative free subsemigroup but also a free algebra in two generators. Moreover, we may assume that all the subrings of our domain are *Ore*.

3. In view of our result, perhaps a first attempt to solve the question would be to consider the polynomial ring in one indeterminate over a noncommutative domain.

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