PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 2, October 1992

ON SOME PROPERTY OF FUNCTIONS DEFINED ON R^2 THAT ARE $\mathscr{I}$-APPROXIMATELY CONTINUOUS WITH RESPECT TO ONE VARIABLE

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(Communicated by Andrew M. Bruckner)

ABSTRACT. Balcerzak, Lazarow, and Wilczyński proved that every separately $\mathscr{I}$-approximately continuous function is Baire 2. In this paper we shall prove that if f is a function $\mathscr{I}$-approximately continuous with respect to one of its variables and of the α-class of Baire with respect to the other one, then f is of the $(\alpha+1)$-class of Baire in R^2.

In this paper $\mathscr{I}$ denotes the σ-ideal of sets of the first category on the line and $\mathscr{B}$ denotes the σ-algebra of subsets of R having the Baire property. We shall say that 0 is an $\mathscr{I}$-density point of a set $A \in \mathscr{B}$ if and only if for each increasing sequence of natural numbers $\{n_m\}_{m\in N}$ there exists a subsequence $\{n_{m_p}\}_{p\in N}$ such that the set $\{x: \chi_{n_{m_p}\cdot A\cap[-1,1]} \nrightarrow 1\} \in \mathscr{I}$, where χ_C denotes the characteristic function of the set C. We shall say that x_0 is an $\mathscr{I}$-density point $A \in \mathscr{B}$ if and only if 0 is an $\mathscr{I}$-density point of the set $A - x_0 = \{x - x_0 : x \in A\}$. We shall say that x_0 is an $\mathscr{I}$-dispersion point of $A \in \mathscr{B}$ if and only if x_0 is an $\mathscr{I}$-density point of $R - A$. The family of all sets $A \in \mathscr{B}$ such that each point of A is its $\mathscr{I}$-density point forms a topology called the $\mathscr{I}$-density topology. The functions that are continuous with respect to the $\mathscr{I}$-density topology are called $\mathscr{I}$-approximately continuous (see [4]).

We shall say that $f: R^2 \to R$ is *$\mathscr{I}$-approximately continuous in the direction x* (resp. y) *at* (x_0, y_0) if the function $f(x, y_0)$ (resp. $f(x_0, y)$) is $\mathscr{I}$-approximately continuous at x_0 (resp. y_0) as a function of x (resp. y). We shall say that $f: R^2 \to R$ is *separately approximately continuous* if and only if f is $\mathscr{I}$-approximately continuous in the direction x and y simultaneously.

In the sequel we shall need the following

Proposition. *Let G be an open set of the real line; then 0 is an $\mathscr{I}$-dispersion point of G if and only if for every natural number n there exist a natural number k and a real number $\delta > 0$ such that for each $h \in (0, \delta)$ and for each $i \in \{1, \dots, n\}$ there exist two natural numbers $j, j' \in \{1, \dots, k\}$ such that*

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk}\right)h, \left(\frac{i-1}{n} + \frac{j}{nk}\right)h\right) = \varnothing$$

Received by the editors February 25, 1991.
1991 *Mathematics Subject Classification*. Primary 26B05, 26A21.

and

$$G \cap \left(-\left(\frac{i-1}{n} + \frac{j'}{nk}\right) h, \left(\frac{i-1}{n} + \frac{j'-1}{nk}\right) h \right) = \varnothing .$$

For the proof see [2, Theorem 1].

We shall say that x_0 is a deep $\mathscr{I}$-density point of the set $A \in \mathscr{B}$ if there exists an open set $B \supset R - A$ such that x_0 is an $\mathscr{I}$-dispersion point of B. The family of all sets $A \in \mathscr{B}$ such that each point of A is its deep $\mathscr{I}$-density point forms a topology called the deep $\mathscr{I}$-density topology. The $\mathscr{I}$-approximately continuous functions are continuous with respect to this topology (see [5, 3]).

Lemma. *Let $f: R^2 \to R$ be a function $\mathscr{I}$-approximately continuous at (x_0, y_0) in the direction x. Then $f(x_0, y_0) \le \alpha$ if and only if for every $n \in N$ there exist $k \in N$, $k > n$, and n rational numbers $r_1^n, \dots, r_n^n$ such that for each $i \in \{1, \dots, n\}$, $r_i^n \in (x_0 + (i-1)/nk, x_0 + i/nk)$, and $f(r_i^n, y_0) < \alpha + 1/n$.*

Proof. For simplicity assume that $(x_0, y_0) = (0, 0)$. Let $n \in N$; if $f(0, 0) \le \alpha$ then $f(0, 0) < \alpha + 1/n$ and $(0, 0)$ is a deep $\mathscr{I}$-density point of the set $\{(x, 0) : f(x, 0) < \alpha + 1/n\}$ in the direction x. Therefore there exist $p \in N$ and $\delta > 0$ such that for each $i \in \{1, \dots, n\}$ and for each $h < \delta$ there exists $j \in \{1, \dots, k\}$ such that

$$\left[\frac{(i-1)p + j - 1}{np} h, \frac{(i-1)p + j}{np} h\right] \times \{0\} \subset \left\{(x, 0) : f(x, 0) < \alpha + \frac{1}{n}\right\} .$$

Let $k > \max\{n, 1/\delta\}$. Then for each $i \in \{1, \dots, n\}$ there exists a rational number

$$r_i^n \in \left(\frac{(i-1)p + j - 1}{np} \frac{1}{k}, \frac{(i-1)p + j}{np} \frac{1}{k}\right) \subset \left[\frac{i-1}{nk}, \frac{i}{nk}\right]$$

such that $f(r_i^n, 0) < \alpha + 1/n$.

Conversely, for each $n \in N$ let k be a natural number greater than n, and let $r_1^n, \dots, r_n^n$ be rational numbers such that for each $i \in \{1, \dots, n\}$ we have $r_i^n \in ((i-1)/nk, i/nk)$ and $f(r_i^n, 0) < \alpha + 1/n$. We shall show that $f(0, 0) \le \alpha$.

Let $A = \bigcup_{n=1}^{\infty} \{r_1^n, \dots, r_n^n\}$. We shall show that for each $p \in N$ and for each $\delta > 0$ there exists $h < \delta$ such that for each $j \in \{1, \dots, p\}$, $((j-1)h/p, jh/p) \cap A \ne \varnothing$. Let $p \in N$ and $\delta > 0$. Choose $n \in N$ such that $n > \max\{2p, \frac{1}{\delta}\}$. Then there exist $k > n$ and $r_1^n, \dots, r_n^n$ such that for each $i \in \{1, \dots, n\}$, $r_i^n \in ((i-1)/nk, i/nk)$. Put $h = 1/k < 1/n < \delta$.

Since $n > 2p$, we observe that for each $j \in \{1, \dots, p\}$ there exists $i \in \{1, \dots, n\}$ such that

$$\left[\frac{i-1}{nk}, \frac{i}{nk}\right] \subset \left(\frac{j-1}{p} \frac{1}{k}, \frac{j}{p} \frac{1}{k}\right) .$$

Therefore for each $j \in \{1, \dots, p\}$,

$$\left(\frac{j-1}{p} \frac{1}{k}, \frac{j}{p} \frac{1}{k}\right) \cap A \ne \varnothing .$$

If $f(0, 0) > \alpha$ then there exists $s \in N$ such that $f(0, 0) > \alpha + 1/s$.

Since f is an $\mathscr{I}$-approximately continuous function in the direction x at $(0, 0)$, then there exists a closed set $F \subset \{(x, 0) : f(x, 0) > \alpha + 1/s\}$ such

that $(0, 0)$ is a point of $\mathscr{I}$-density of F in the direction x. Thus there exist $p_0 \in N$ and $\delta > 0$ such that for each $h < \delta$ there exists $j \in \{1, \dots, p_0\}$ such that $[(j-1)h/p_0, jh/p_0] \times \{0\} \subset F$.

Let $h_1 < \delta$ such that for each $j \in \{1, \dots, p_0\}$, $((j-1)h_1/p_0, jh_1/p_0) \cap A \neq \varnothing$.

Then there exists $r_1 \in N$ such that for each $j \in \{1, \dots, p_0\}$ we have

$$\left(\frac{j-1}{p_0}h_1, \frac{j}{p_0}h_1\right) \cap \{r_1^{n_1}, \dots, r_{n_1}^{n_1}\} \neq \varnothing.$$

Now let $h_2 < r_1$ such that for each $j \in \{1, \dots, p_0\}$ we have $((j-1)h_2/p_0, jh_2/p_0) \cap A \neq \varnothing$. Then there exists $n_2 > n_1$, $n_2 \in N$, such that for each $j \in \{1, \dots, p_0\}$,

$$\left(\frac{j-1}{p_0}h_2, \frac{j}{p_0}h_2\right) \cap \{r_1^{n_2}, \dots, r_{n_2}^{n_2}\} \neq \varnothing.$$

By induction we define sequences $\{h_m\}_{m \in N}$ and $\{n_m\}_{m \in N}$ such that for each $m \in N$ we have $h_m < \delta$,

$$\left(\frac{j-1}{p_0}h_m, \frac{j}{p_0}h_m\right) \cap \{r_1^{n_m}, \dots, r_{n_m}^{n_m}\} \neq \varnothing$$

and $\lim_m h_m = 0$, $\lim_m n_m = \infty$.

Thus, if m_0 is such that $n_{m_0} > s$ then for each $j \in \{1, \dots, n_{m_0}\}$ we have $f(r_i^{n_{m_0}}, 0) < \alpha + 1/n_{m_0} < \alpha + 1/s$. Therefore for each $j \in \{1, \dots, p_0\}$,

$$\left(\frac{j-1}{p_0}h_{m_0}, \frac{j}{p_0}h_{m_0}\right) \cap \left\{(x, 0) : f(x, 0) < \alpha + \frac{1}{s}\right\} \neq \varnothing,$$

a contradiction.

Then $f(0, 0) \leq \alpha$ and the proof is completed.

An analogous proposition obviously holds for $(x_0, y_0) \in R^2$ such that $f(x_0, y_0) \geq \alpha$ and for f $\mathscr{I}$-approximately continuous at (x_0, y_0) in the direction y.

Theorem. *Let $f: R^2 \to R$ be a function such that for each $(x_0, y_0) \in R^2$, f is $\mathscr{I}$-approximately continuous at (x_0, y_0) in the direction x and of the Baire class α (with respect to the natural topology) in the direction y. Then f is a function of the Baire class $\alpha + 1$ on the plane.*

Proof. By the previous lemma we have that for each $\lambda \in R$

$$\begin{aligned}
&\{(x, y) : f(x, y) \leq \lambda\} \\
&= \bigcap_{n=1}^{\infty} \bigcup_{k=n+1}^{\infty} \bigcup_{\{r_1^n, \dots, r_n^n\} \in Q} \bigcap_{i=1}^{n} \left(\left\{x : \frac{i-1}{nk} < |x - r_i^n| < \frac{i}{nk}\right\} \times R\right) \\
&\qquad \cap \left(R \times \left\{y : f(r_i^n, y) < \lambda + \frac{1}{n}\right\}\right),
\end{aligned}$$

where Q denotes the set of rational numbers.

If f is of the Baire class α in the direction y at every $(x, y) \in R^2$ then the set $\{y : f(r_i^n, y) < \lambda + 1/n\}$ belongs to F_α if α is odd and to G_α if α is even.[1]

[1]For each ordinal α such that $0 < \alpha < \omega_1$, F_α and G_α denote the Borel classes with respect to the natural topology.

Therefore $\{(x, y) : f(x, y) \leq \lambda\}$ belongs to $F_{\alpha+1}$ if α is odd and to $G_{\alpha+1}$ if α is even.

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DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "R. CACCIOPPOLI" VIA CINTIA NAPOLI, ITALY

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