ON SOME PROPERTY OF FUNCTIONS DEFINED ON $R^2$ THAT ARE $\mathcal{J}$-APPROXIMATELY CONTINUOUS WITH RESPECT TO ONE VARIABLE

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Abstract. Balcerzak, Lazarow, and Wilczyński proved that every separately $\mathcal{J}$-approximately continuous function is Baire 2. In this paper we shall prove that if $f$ is a function $\mathcal{J}$-approximately continuous with respect to one of its variables and of the $\alpha$-class of Baire with respect to the other one, then $f$ is of the $(\alpha + 1)$-class of Baire in $R^2$.

In this paper $\mathcal{I}$ denotes the $\sigma$-ideal of sets of the first category on the line and $\mathcal{B}$ denotes the $\sigma$-algebra of subsets of $R$ having the Baire property. We shall say that $0$ is an $\mathcal{I}$-density point of a set $A \in \mathcal{B}$ if and only if for each increasing sequence of natural numbers $\{n_m\}_{m \in \mathbb{N}}$ there exists a subsequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that the set $\{x : \chi_{n_{m_p} \cdot A \cap [1, -1]} \rightarrow 1 \} \in \mathcal{I}$, where $\chi_C$ denotes the characteristic function of the set $C$. We shall say that $x_0$ is an $\mathcal{I}$-density point $A \in \mathcal{B}$ if and only if $x_0$ is an $\mathcal{I}$-density point of the set $A - x_0 = \{x - x_0 : x \in A\}$. We shall say that $x_0$ is an $\mathcal{I}$-dispersion point of $A \in \mathcal{B}$ if and only if $x_0$ is an $\mathcal{I}$-density point of $R - A$. The family of all sets $A \in \mathcal{B}$ such that each point of $A$ is its $\mathcal{I}$-density point forms a topology called the $\mathcal{I}$-density topology. The functions that are continuous with respect to the $\mathcal{I}$-density topology are called $\mathcal{I}$-approximately continuous (see [4]).

We shall say that $f : R^2 \rightarrow R$ is $\mathcal{I}$-approximately continuous in the direction $x$ (resp. $y$) at $(x_0, y_0)$ if the function $f(x, y_0)$ (resp. $f(x_0, y)$) is $\mathcal{I}$-approximately continuous at $x_0$ (resp. $y_0$) as a function of $x$ (resp. $y$). We shall say that $f : R^2 \rightarrow R$ is separately approximately continuous if and only if $f$ is $\mathcal{I}$-approximately continuous in the direction $x$ and $y$ simultaneously.

In the sequel we shall need the following

Proposition. Let $G$ be an open set of the real line; then $0$ is an $\mathcal{I}$-dispersion point of $G$ if and only if for every natural number $n$ there exist a natural number $k$ and a real number $\delta > 0$ such that for each $h \in (0, \delta)$ and for each $i \in \{1, \ldots, n\}$ there exist two natural numbers $j, j' \in \{1, \ldots, k\}$ such that

$$G \cap \left( \left( \frac{i - 1}{n} + \frac{j - 1}{nk} \right) h, \left( \frac{i - 1}{n} + \frac{j}{nk} \right) h \right) = \emptyset$$

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and
\[ G \cap \left( -\left( \frac{i-1}{n} + \frac{j}{nk} \right) h, \left( \frac{i-1}{n} + \frac{j-1}{nk} \right) h \right) = \emptyset. \]

For the proof see [2, Theorem 1].

We shall say that \( x_0 \) is a deep \( \mathcal{I} \)-density point of the set \( A \in \mathcal{B} \) if there exists an open set \( B \supset R - A \) such that \( x_0 \) is an \( \mathcal{I} \)-dispersion point of \( B \). The family of all sets \( A \in \mathcal{B} \) such that each point of \( A \) is its deep \( \mathcal{I} \)-density point forms a topology called the deep \( \mathcal{I} \)-density topology. The \( \mathcal{I} \)-approximately continuous functions are continuous with respect to this topology (see [5, 3]).

**Lemma.** Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function \( \mathcal{I} \)-approximately continuous at \( (x_0, y_0) \) in the direction \( x \). Then \( f(x_0, y_0) \leq \alpha \) if and only if for every \( n \in \mathbb{N} \) there exist \( k \in \mathbb{N} \), \( k > n \), and \( n \) rational numbers \( r^n_1, \ldots, r^n_n \) such that for each \( i \in \{1, \ldots, n\} \), \( r^n_i \in (x_0 + (i-1)/nk, x_0 + i/nk) \), and \( f(r^n_i, y_0) < \alpha + 1/n \).

**Proof.** For simplicity assume that \( (x_0, y_0) = (0,0) \). Let \( n \in \mathbb{N} \); if \( f(0,0) \leq \alpha \) then \( f(0,0) < \alpha + 1/n \) and \( (0,0) \) is a deep \( \mathcal{I} \)-density point of the set \( \{ (x,0) : f(x,0) < \alpha + 1/n \} \) in the direction \( x \). Therefore there exist \( p \in \mathbb{N} \) and \( \delta > 0 \) such that for each \( i \in \{1, \ldots, n\} \) and for each \( h < \delta \) there exists \( j \in \{1, \ldots, k\} \) such that
\[
\left[ \frac{(i-1)p + j - 1}{np}, \frac{(i-1)p + j}{np} \right] \times \{0\} \subset \left\{ (x,0) : f(x,0) < \alpha + \frac{1}{n} \right\}.
\]

Let \( k > \max\{n, 1/\delta\} \). Then for each \( i \in \{1, \ldots, n\} \) there exists a rational number \( r^n_i \in \left[ \frac{(i-1)p + j - 1}{np}, \frac{(i-1)p + j}{np} \right] \subset \left[ \frac{i-1}{nk}, \frac{i}{nk} \right] \) such that \( f(r^n_i, 0) < \alpha + 1/n \).

Conversely, for each \( n \in \mathbb{N} \) let \( k \) be a natural number greater than \( n \), and let \( r^n_1, \ldots, r^n_n \) be rational numbers such that for each \( i \in \{1, \ldots, n\} \) we have \( r^n_i \in ((i-1)/nk, i/nk) \) and \( f(r^n_i, 0) < \alpha + 1/n \). We shall show that \( f(0,0) \leq \alpha \).

Let \( A = \bigcup_{n=1}^{\infty} \{ r^n_1, \ldots, r^n_n \} \). We shall show that for each \( p \in \mathbb{N} \) and \( \delta > 0 \) there exists \( h < \delta \) such that for each \( j \in \{1, \ldots, p\} \), \( ((j-1)h/p, jh/p) \cap A \neq \emptyset \). Let \( p \in \mathbb{N} \) and \( \delta > 0 \). Choose \( n \in \mathbb{N} \) such that \( n > \max\{2p, \frac{1}{\delta}\} \). Then there exist \( k > n \) and \( r^n_1, \ldots, r^n_n \) such that for each \( i \in \{1, \ldots, n\} \), \( r^n_i \in ((i-1)/nk, i/nk) \). Put \( h = 1/k < 1/n < \delta \).

Since \( n > 2p \), we observe that for each \( j \in \{1, \ldots, p\} \) there exists \( i \in \{1, \ldots, n\} \) such that
\[
\left[ \frac{i-1}{nk}, \frac{i}{nk} \right] \subset \left[ \frac{j-1}{p}, \frac{j}{p} \right].
\]

Therefore for each \( j \in \{1, \ldots, p\} \),
\[
\left( \frac{j-1}{p}, \frac{j}{p} \right) \cap A \neq \emptyset.
\]

If \( f(0,0) > \alpha \) then there exists \( s \in \mathbb{N} \) such that \( f(0,0) > \alpha + 1/s \).

Since \( f \) is an \( \mathcal{I} \)-approximately continuous function in the direction \( x \) at \( (0,0) \), then there exists a closed set \( F \subset \{(x,0) : f(x,0) > \alpha + 1/s\} \) such
that \((0, 0)\) is a point of \(\mathcal{F}\)-density of \(F\) in the direction \(x\). Thus there exist \(p_0 \in \mathbb{N}\) and \(\delta > 0\) such that for each \(h < \delta\) there exists \(j \in \{1, \ldots, p_0\}\) such that \([(j - 1)h/p_0, jh/p_0] \times \{0\} \subset F\).

Let \(h_1 < \delta\) such that for each \(j \in \{1, \ldots, p_0\}\), \([(j - 1)h_1/p_0, jh_1/p_0] \cap A \neq \emptyset\).

Then there exists \(r_1 \in \mathbb{N}\) such that for each \(j \in \{1, \ldots, p_0\}\) we have
\[
\left(\frac{j - 1}{p_0} h_1, \frac{j}{p_0} h_1\right) \cap \{r_1^n, \ldots, r_{n_1}^n\} \neq \emptyset.
\]

Now let \(h_2 < r_1\) such that for each \(j \in \{1, \ldots, p_0\}\) we have \([(j - 1)h_2/p_0, jh_2/p_0] \cap A \neq \emptyset\). Then there exists \(n_2 > n_1, n_2 \in \mathbb{N}\), such that for each \(j \in \{1, \ldots, p_0\}\),
\[
\left(\frac{j - 1}{p_0} h_2, \frac{j}{p_0} h_2\right) \cap \{r_1^{n_2}, \ldots, r_{n_2}^{n_2}\} \neq \emptyset.
\]

By induction we define sequences \(\{h_m\}_{m \in \mathbb{N}}\) and \(\{n_m\}_{m \in \mathbb{N}}\) such that for each \(m \in \mathbb{N}\) we have \(h_m < \delta\),
\[
\left(\frac{j - 1}{p_0} h_m, \frac{j}{p_0} h_m\right) \cap \{r_1^{n_m}, \ldots, r_{n_m}^{n_m}\} \neq \emptyset
\]
and \(\lim_m h_m = 0\), \(\lim_m n_m = \infty\).

Thus, if \(m_0\) is such that \(n_{m_0} > s\) then for each \(j \in \{1, \ldots, n_{m_0}\}\) we have \(f(r_1^{n_{m_0}}, 0) < \alpha + 1/n_{m_0} < \alpha + 1/s\). Therefore for each \(j \in \{1, \ldots, p_0\}\),
\[
\left(\frac{j - 1}{p_0} h_{m_0}, \frac{j}{p_0} h_{m_0}\right) \cap \left\{(x, 0) : f(x, 0) < \alpha + \frac{1}{s}\right\} \neq \emptyset,
\]
a contradiction.

Then \(f(0, 0) \leq \alpha \) and the proof is completed.

An analogous proposition obviously holds for \((x_0, y_0) \in \mathbb{R}^2\) such that \(f(x_0, y_0) \geq \alpha\) and for \(f\ \mathcal{F}\)-approximately continuous at \((x_0, y_0)\) in the direction \(y\).

**Theorem.** Let \(f: \mathbb{R}^2 \to \mathbb{R}\) be a function such that for each \((x_0, y_0) \in \mathbb{R}^2\), \(f\) is \(\mathcal{F}\)-approximately continuous at \((x_0, y_0)\) in the direction \(x\) and of the Baire class \(\alpha\) (with respect to the natural topology) in the direction \(y\). Then \(f\) is a function of the Baire class \(\alpha + 1\) on the plane.

**Proof.** By the previous lemma we have that for each \(\lambda \in \mathbb{R}\)
\[
\{(x, y) : f(x, y) \leq \lambda\}
\]

\[
= \bigcap_{n=1}^{\infty} \bigcup_{k=n+1}^{\infty} \bigcup_{\{r_1^n, \ldots, r_{n}^n\} \in Q} \bigcap_{i=1}^{n} \left\{x : \left|\frac{i - 1}{n k} < |x - r_i^n| < \frac{i}{n k}\right\} \times \mathbb{R}
\]

\[
\cap \left(\mathbb{R} \times \left\{y : f(r_1^n, y) < \lambda + \frac{1}{n}\right\}\right),
\]

where \(Q\) denotes the set of rational numbers.

If \(f\) is of the Baire class \(\alpha\) in the direction \(y\) at every \((x, y) \in \mathbb{R}^2\) then the set \(\{y : f(r_1^n, y) < \lambda + 1/n\}\) belongs to \(F_\alpha\) if \(\alpha\) is odd and to \(G_\alpha\) if \(\alpha\) is even.\(^1\)

\(^1\)For each ordinal \(\alpha\) such that \(0 < \alpha < \omega_1\), \(F_\alpha\) and \(G_\alpha\) denote the Borel classes with respect to the natural topology.
Therefore \( \{(x, y) : f(x, y) \leq \lambda\} \) belongs to \( F_{\alpha+1} \) if \( \alpha \) is odd and to \( G_{\alpha+1} \) if \( \alpha \) is even.

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