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# COMPLEX SEQUENCES WHOSE "MOMENTS" ALL VANISH

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*Dedicated to Stephen Puckette*

ABSTRACT. Must a sequence $\{z_k\}$ of complex numbers be identically zero if $\sum f(z_k) = 0$ for every entire function $f$ vanishing at the origin? Lenard's example of a nonzero sequence of complex numbers whose power sums ("moments") all vanish is shown to give a negative answer to this question and to lead to a novel representation theorem for entire functions.

On the positive side it is proved that if $\{z_k\}$ is in $l^p$ where $p < \infty$, then vanishing moments imply $\{z_k\}$ is identically zero. Virtually the same proof shows that, on a Hilbert space, two compact normal operators $A$ and $B$ with trivial kernels are unitarily equivalent if some power of each belongs to the trace class and $\operatorname{tr}(A^n) = \operatorname{tr}(B^n)$ for all $n$ in a set of positive integers with asymptotic density one.

Consider a sequence $z_1, z_2, z_3, \ldots$ of complex numbers with the property that $\sum f(z_k) = 0$ for every entire function $f$ vanishing at the origin. A finite sequence with this property must be identically zero. *Reason*: If the set $\{z_k\}$ is finite and contains a nonzero number $w$, then $\sum f(z_k) \neq 0$ if $f(z) = zp(z)$, where $p$ is a polynomial vanishing at all $z_k$ distinct from $w$ but not at $w$.

What about the general case?

**Conjecture.** *Any sequence with the above property must be identically zero.*

We investigate this conjecture here. Theorem 2 below shows that it is false, and yields, as a byproduct in Corollary 3, a novel way to represent entire functions. Corollary 6 shows that the conjecture is true if the sequence is in $l^p$, where $p < \infty$. This also has a byproduct. Theorem 7 characterizes unitary equivalence within certain classes of normal (*respectively, positive*) compact operators $A$ and $B$ by agreement of trace$(A^n)$ and trace$(B^n)$ for "essentially all" (*respectively, infinitely many*) positive integers $n$.

## 1. The conjecture is false

Andrew Lenard [3] has given an example of a sequence $\{\lambda_k\}$ that begins

(1) $$1, -1, i/2, -i/2, i/2, -i/2, i/2, -i/2, i/2, -i/2$$

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and continues in a beautiful pattern (to be outlined in the proof of Theorem 2) in such a way that all its power sums vanish. That is,

$$0=\sum \lambda_k=\sum(\lambda_k)^2=\sum(\lambda_k)^3=\cdots .$$

Here we shall refer to such power sums as "moments." If all the moments of $\{z_k\}$ vanish then, clearly, so do all the moments of the sequence $\{p(z_k)\}$, where $p$ is any polynomial without constant term. This raises the question whether the vanishing-moments property is preserved under composition with more general functions. We first show that for any vanishing-moments sequence satisfying the mild condition (3) given below, all the moments of the sequence $\{f(z_k)\}$ vanish for any function $f$ that is analytic in a sufficiently large region, provided $f(0)=0$.

**Lemma 1.** *Let $\{z_k\}$ be a sequence of complex numbers satisfying*

$$\text{(2)} \qquad 0=\sum_{k=1}^{\infty} z_k=\sum_{k=1}^{\infty}(z_k)^2=\sum_{k=1}^{\infty}(z_k)^3=\cdots ,$$

*and define an associated sequence $\{M_p\}$ of real numbers by*

$$M_1=\max_n\left|\sum_{k=1}^{n} z_k\right|, \qquad M_2=\max_n\left|\sum_{k=1}^{n}(z_k)^2\right|, \qquad \textit{etc.}$$

*Assume that for some constant $C$ and some integer $N_0$,*

$$\text{(3)} \qquad p>N_0 \quad \textit{implies} \quad M_p\le C^p.$$

*Let $f$ be an entire function (or, more generally, let $f$ be analytic in $\{z \mid |z|<R\}$, where $R$ exceeds both $C$ and $\max|z_k|$), and assume that $f$ vanishes at the origin. Then $\sum_{k=1}^{\infty} f(z_k)=0$.*

*Proof.* Given $\varepsilon>0$, choose $N\ge N_0$ such that $\sum_{p=N+1}^{\infty}|a_p|C^p<\varepsilon$, where $\sum_{p=1}^{\infty} a_p z^p$ is the series representation of $f$. Any partial sum of $\sum f(z_k)$ may be expressed as

$$\sum_{k=1}^{K} f(z_k)=a_1\sum_{k=1}^{K} z_k+a_2\sum_{k=1}^{K}(z_k)^2+\cdots+a_N\sum_{k=1}^{K}(z_k)^N+B,$$

where $|B|\le\sum_{p=N+1}^{\infty}|a_p|M_p<\varepsilon$ by (3). Hence, we have

$$\limsup_{K\to\infty}\left|\sum_{k=1}^{K} f(z_k)\right|<\limsup_{K\to\infty}\left|a_1\sum_{k=1}^{K} z_k+a_2\sum_{k=1}^{K} z_k^2+\cdots+a_N\sum_{k=1}^{K} z_k^N\right|+\varepsilon.$$

But the lim sup on the right-hand side is zero by (2) and, therefore,

$$\limsup_{K\to\infty}\left|\sum_{k=1}^{K} f(z_k)\right|<\varepsilon. \quad \square$$

By applying Lemma 1 to $f^p$ instead of $f$, we see that $\sum f(z_k)^p=0$. Thus the sequence $\{f(z_k)\}$ in Lemma 1 inherits the vanishing-moments property (2) of $\{z_k\}$.

**Theorem 2.** *Lenard's sequence* $\{\lambda_k\}$ *has the property that* $\sum f(\lambda_k) = 0$ *for any entire function* $f$ *vanishing at the origin.*

*Proof*. We must first recall Lenard's construction that defines inductively longer and longer initial segments of $\{\lambda_k\}$. Let $\varepsilon_p = \exp(i\pi/p)/p$ and let $s_0$ denote the sequence of length 1 whose sole member is 1. Let $s_1$ consist of the sequence $s_0$ followed by the sequence obtained from $s_0$ by multiplying it by $\varepsilon_1$. Let $s_2$ be $s_1$ followed by $2^2$ copies of the sequence obtained from $s_1$ by multiplying $s_1$ by $\varepsilon_2$. (Thus $s_2$ is the ten-term sequence displayed in (1).) Let $s_3$ be $s_2$ followed by $3^3$ copies of the sequence obtained from $s_2$ by multiplying $s_2$ by $\varepsilon_3$. And so on. Lenard [3] proves that the infinite sequence $\{\lambda_k\}$ defined inductively by this process satisfies condition (2). The verification of condition (3) will then show that the theorem follows from Lemma 1.

To see that condition (3) is satisfied, note first that the moduli of the partial sums of the $p$th powers of $\{\lambda_k\}$, which tend to zero, have passed their maximum value before the $p$th stage $s_p$ has been completed. *Reason*: As Lenard points out, his entire sequence may be viewed as a succession of infinitely many copies of $s_p$ scaled with complex factors whose moduli tend to zero, and $s_p$ is constructed so that the sum of its $p$th powers vanishes. Thus each partial sum of the $p$th powers of $\{\lambda_k\}$, regardless of the number of summands, will be equal to some partial sum of the $p$th powers of $s_p$ multiplied by a factor of modulus not exceeding 1.

Therefore, in calculating the $p$th term of the associated $M$-sequence given by

$$M_p = \max_n \left|\sum_{k=1}^{n} (\lambda_k)^p\right|,$$

we need not consider values of $n$ larger than the number of members constructed at the $p$th stage, which number we shall denote by $|s_p|$. Thus $|s_0| = 1$, $|s_1| = 2$, $|s_2| = 10$, and, in general, $|s_j| = (1 + j^j)|s_{j-1}|$. A generous overestimate of $M_p$ is then given by

$$M_p \leq \sum_{k=1}^{|s_p|} |\lambda_k|^p = \|s_p\|_p^p,$$

the right-hand side denoting the $p$th power of the $l^p$-norm of $s_p$.

We now verify condition (3) by showing $\|s_p\|_p^p \leq 2^p$. It is clear from the construction of $s_p$ that

$$\|s_p\|_p^p = \|s_{p-1}\|_p^p + p^p|\varepsilon_p|^p\|s_{p-1}\|_p^p = 2\|s_{p-1}\|_p^p \leq 2\|s_{p-1}\|_{p-1}^{p-1},$$

the inequality holding since no element of Lenard's sequence has a modulus exceeding 1. Thus, $\|s_p\|_p^p \leq 2\|s_{p-1}\|_{p-1}^{p-1}$ for all $p \geq 2$, and this, coupled with the fact that $\|s_1\|_1^1 = 2$, implies that $\|s_p\|_p^p \leq 2^p$. □

The following corollary shows that if $g$ is an entire function, then $g(z)$ may be represented as the sum of a (conditionally convergent) series involving only values of $g$ taken in a ball of radius $|z|\varepsilon$ centered at the origin, where $\varepsilon$ may be arbitrarily small.

**Corollary 3.** *If $\varepsilon > 0$, there exits a sequence $\{\mu_k\}_{k=2}^{\infty}$ with $|\mu_k| < \varepsilon$ such that for any entire function $g$,*

$$g(z) = g(0) + \sum_{k=2}^{\infty}(g(0) - g(\mu_k z)). \tag{4}$$

*Proof.* Given $\varepsilon > 0$, choose $n$ large enough to ensure $|p_n(\lambda_k)| < \varepsilon$ for all $k > 1$, where $p_n$ is the polynomial given by $p_n(w) = (2^n - 1)^{-1}[(w+1)^n - 1]$. For fixed $z$, let the entire function $f$ be defined by $f(w) = g(zp_n(w)) - g(0)$. Applying Theorem 2 to $f$ yields

$$0 = \sum_{k=1}^{\infty} f(\lambda_k) = f(1) + \sum_{k=2}^{\infty} f(\lambda_k),$$

since $\lambda_1 = 1$. Set $\mu_k$ equal to $p_n(\lambda_k)$ to obtain the representation (4). □

If $g(z)$ is entire, then $zg'(z) = -\sum \mu_k zg'(\mu_k z)$ by Corollary 3 applied to $zg'(z)$. It follows that $g'(z) = -\sum \mu_k g'(\mu_k z)$, justifying termwise differentiation of (4).

**Corollary 4.** *Let $U$ be an open set containing the origin. Then $U$ contains a nonzero complex sequence $\{v_k\}$ such that $\sum f(v_k) = 0$ for every function $f$ analytic on $U$ and vanishing at the origin.*

*Proof.* Let $\varepsilon$ be the radius of some neighborhood of the origin contained in $U$. We have seen that Lenard's vanishing-moments sequence $\{\lambda_k\}$ is associated with an $M$-sequence bounded by $2^p$, so the sequence defined by $v_k = (\varepsilon/2)\lambda_k$ is a vanishing-moments sequence lying within this $\varepsilon$-neighborhood of zero and associated with an $M$-sequence bounded by $\varepsilon^p$. By Lemma 1, the sequence $\{v_k\}$ has the required property. □

## 2. The conjecture is true in $l^p$ if $p < \infty$

We shall prove a theorem about rearrangements of nonzero sequences from which it trivially follows that $l^p$, where $p < \infty$, contains no nonzero sequences with the vanishing-moments property. The theorem will be useful also in characterizing unitary equivalence within a certain class of normal compact operators. Recall first that in the case of a convergent sequence, its limit and its Césàro limit agree, where by the Césàro limit of a sequence $\{s_n\}$ is meant the limit of the sequence of successive averages: $s_1$, $(s_1 + s_2)/2$, $(s_1 + s_2 + s_3)/3$, etc.

**Theorem 5.** *Let $\{\alpha_k\}$ and $\{\beta_k\}$ be sequences of nonzero complex numbers belonging to $l^p$ with $p < \infty$, and assume that their sums of powers agree for each positive integer $n \geq p$,*

$$\sum_{k=1}^{\infty}(\alpha_k)^n = \sum_{k=1}^{\infty}(\beta_k)^n. \tag{5}$$

*Then each of the sequences is a rearrangement of the other.*

*Proof*. First consider a sequence $\{w^k\}$, where $|w| = 1$ but $w \neq 1$. The sequence cannot converge, but continually winds around the unit circle in the complex plane. However, its Césàro limit is zero,

$$\frac{1}{m}\sum_{k=1}^{m} w^k \to 0, \tag{6}$$

as is easily seen from the formula for the sum of a finite geometric series.

We first show $A = B$, where $A = \max|\alpha_k|$ and $B = \max|\beta_k|$. Suppose $A > B$. Assumption (5) implies $\sum(\alpha_k/A)^n = \sum(\beta_k/A)^n$ for each $n \geq p$, and since $|\beta_k/A| \leq B/A < 1$ for all $k$, the right-hand side of this equation tends to zero as $n \to \infty$. (A heavy-handed but quick way to see this is to use the dominated convergence theorem with $\{|\beta_k/A|^p\}$ as majorant: $\sum(\beta_k/A)^n \to \sum 0 = 0$.) Thus Limit $\sum(\alpha_k/A)^n = 0$. Let $z_1, z_2, \ldots, z_s$ denote the (finite number of) terms $\alpha_k/A$ where $|\alpha_k| = A$. It is obvious that

$$\sum_{j=1}^{s}(z_j)^n = \sum_{k=1}^{\infty}(\alpha_k/A)^n - \sum_{|\alpha_k|<A}(\alpha_k/A)^n,$$

and, as we have just seen, the first summation on the right-hand side tends to zero as $n \to \infty$. So does the second (dominated convergence with $\{|\alpha_k/A|^p\}$ as majorant), since $(\alpha_k/A)^n \to 0$ for each index $k$ of this summation. Therefore, $\sum(z_j)^n$ tends to zero, where each $z_j$ is a complex number of unit modulus.

It is quite apparent that this is an impossibility, but to nail down the contradiction, let $w_j = z_j/z_1$ (so that $w_1 = 1$) and observe that $\sum(w_j)^n$ must tend to zero as $n \to \infty$. This implies

$$\operatorname*{Limit}_{n\to\infty}\sum_{w_j\neq 1}(w_j)^n = -\operatorname*{Limit}_{n\to\infty}\sum_{w_j=1}(w_j)^n = -\sum_{w_j=1} 1,$$

showing that the limit of $s_n$, where $s_n = \sum_{w_j\neq 1}(w_j)^n$, exists and is equal to some negative integer. The Césàro limit must then be this same negative integer. But finitely many applications of (6), taking $w = w_j$ for each $w_j \neq 1$, show that the Césàro limit of the sequence $\{s_n\}$ is zero. Thus the assumption $A > B$ (and, by a symmetric argument, the assumption $B < A$) leads to a contradiction. Hence, $A = B$.

Now consider any complex number $Ae^{i\theta}$ of modulus $A$. We claim that the $\alpha_k$'s and the $\beta_k$'s hit $Ae^{i\theta}$ the same number of times. To see this, note that equations (5) imply

$$\sum_{k=1}^{\infty}(\alpha_k/Ae^{i\theta})^n = \sum_{k=1}^{\infty}(\beta_k/Ae^{i\theta})^n \tag{7}$$

for all $n \geq p$. The left-hand side of equation (7) may be written as

$$\sum_{|\alpha_k|<A}(\alpha_k/Ae^{i\theta})^n + \sum_{|\alpha_k|=A}(\alpha_k/Ae^{i\theta})^n$$

where the first term here tends to zero as $n \to \infty$, while the second term, by (6), has a Césàro limit equal to the cardinality of the set $\{k | \alpha_k = Ae^{i\theta}\}$. Hence the Césàro limit of the left-hand side of (7) is a nonnegative integer equal to the number of times that the sequence $\{\alpha_k\}$ hits $Ae^{i\theta}$. By a similar argument, relying on the fact that $A = B$, the Césàro limit of the right-hand side of (7) is equal to the number of times the sequence $\{\beta_k\}$ hits $Ae^{i\theta}$. Since the Césàro limits of both sides of (7) must agree, our claim is proved, making it evident that the $\alpha$'s of maximum modulus may be rearranged to match the $\beta$'s of maximum modulus.

Equations (5) will hold if each of the $\alpha$'s and $\beta$'s of maximum modulus is replaced by a zero, so a rearrangement of the $\alpha$'s and $\beta$'s of next largest modulus can be effected by the same process. All the (original) $\alpha$'s and $\beta$'s are assumed nonzero and, therefore, continuation of this peeling-off process can be seen to implement an inductive definition of a permutation $\sigma$ of the positive integers such that $\alpha_k = \beta_{\sigma(k)}$. □

Lenard's example shows that convergence of all power sums of a sequence does not imply that the sequence belongs to $l^p$ for some $p < \infty$. (A simpler example of this is given by $\alpha_k = \exp(ik)/\ln(k+1)$. For each $p$, $\sum(\alpha_k)^p$ converges but $\sum |\alpha_k|^p$ does not.)

There are no nonzero sequences with vanishing moments in $l^p$ if $p < \infty$, as seen by the following corollary that says a bit more.

**Corollary 6.** *Assume* $\{\gamma_k\}$ *is in* $l^p$ *with* $p < \infty$ *and*

$$0 = \sum_{k=1}^{\infty} (\gamma_k)^p = \sum_{k=1}^{\infty} (\gamma_k)^{2p} = \sum_{k=1}^{\infty} (\gamma_k)^{3p} = \cdots .$$

*Then* $\gamma_k = 0$ *for all* $k$.

*Proof.* If not, then a counterexample to Theorem 5 could be obtained by intertwining the (nonzero members of the) sequence $\{\gamma_k\}$ with any $l^p$ sequence $\{\beta_k\}$ of nonzero complex numbers: The sequences $\beta_1^p, \beta_2^p, \beta_3^p, \beta_4^p, \ldots$ and $\beta_1^p, \gamma_1^p, \beta_2^p, \gamma_2^p, \ldots$ are nonzero sequences lying in $l^1$, have the same moments, yet cannot be rearrangements of each other. □

The assertions in Theorem 5 and Corollary 6 can be strengthened. It is not necessary to assume that equations (5) hold for every $n$ in a final segment of integers, but only, as we shall see in the next section, for $n$ belonging to a set of positive integers of asymptotic density one. The motivation for seeking such a generalization comes from operator theory.

## 3. Unitary equivalence of compact normal or positive operators

The *trace class* of a Hilbert space $H$ with inner product $\langle\ ,\ \rangle$ consists of all compact linear operators $T: H \to H$ such that the series $\sum \langle |T| v_j, v_j \rangle$ is convergent, where $\{v_j\}$ is an orthonormal basis of $H$ and $|T| = (T^*T)^{1/2}$. Suppose $A$ and $B$ are normal operators whose $p$th powers lie in the trace class and that $p, p+1, p+2, \ldots$ are solutions to the equation

$$\text{(8)} \qquad \operatorname{tr}(A^\lambda) = \operatorname{tr}(B^\lambda).$$

Does it follow that $A$ and $B$ are unitarily equivalent?

By Theorem 5, since the sequence of eigenvalues of each operator lies in $l^p$, it follows that $A$ and $B$ have the same nonzero eigenvalues occurring with the same multiplicities. Unitary equivalence then clearly follows if the kernels of $A$ and $B$ have the same dimension. This result, as pointed out by the referee, overlaps with special cases of deeper theorems proved in [1, 2] by Deckard and Pearcy. It was curiosity about this question, however, that originally motivated this paper.

The ideas of the preceding section may be modified slightly to yield a stronger result characterizing unitary equivalence of certain compact normal operators. Another modification yields an analogous result for compact positive operators.

**Theorem 7.** *Let $A$ and $B$ be compact operators acting on a Hilbert space $H$. Assume that the kernels of $A$ and $B$ have the same dimension and that some power of each operator lies in the trace class of $H$. Then $A$ and $B$ are unitarily equivalent on $H$ if either*

(a) *$A$ and $B$ are normal operators and equation* (8) *admits as solutions a strictly increasing infinite sequence $\{n_k\}$ of positive integers of asymptotic density one, i.e., satisfying $k/n_k \to 1$ as $k \to \infty$; or*

(b) *$A$ and $B$ are positive operators and equation* (8) *admits as solutions a strictly increasing infinite sequence $\{\lambda_k\}$ of positive real numbers.*

*Proof.* (a) Consider a sequence $\{w^{n_k}\}$, where $|w| = 1$ but $w \neq 1$. The assumption $m/n_m \to 1$ as $m \to \infty$ implies that the Césàro limit of this sequence is zero. To see this, consider taking the limit as $m \to \infty$ of

$$\frac{1}{m}\sum_{k=1}^{m} w^{n_k} = \frac{1}{m}\sum_{k=1}^{n_m} w^k - \frac{1}{m}\sum w^k,$$

where in the last summation the index $k$ runs over the $n_m - m$ positive integers in the set $\{1, 2, \dots, n_m\}$ that are not included in $\{n_1, n_2, \dots, n_m\}$. Each of the two expressions on the right-hand side tends to zero as $m \to \infty$, for the first involves an easily handled geometric series with $w \neq 1$ and the second (since $w$ has unit modulus) is in modulus bounded by $(n_m - m)/m = n_m/m - 1$, which tends to zero by virtue of $\{n_m\}$ having asymptotic density one.

Since an analogue of condition (6) holds for the sequence $\{n_k\}$, it is easy to see that the conclusion of Theorem 5 follows when equations (5) are assumed to hold not for all $n \geq p$, but only for members of a set $\{n_k\}$ having asymptotic density one in the set of positive integers. Making only minor changes in the proof of Theorem 5, we can proceed from this weaker assumption to the same conclusion, then apply the results to the sequence of eigenvalues of $A$ and of $B$. We then see via the spectral theorem that $A$ and $B$ are unitarily equivalent if their kernels have the same dimension.

(b) If $\{\alpha_k\}$ and $\{\beta_k\}$ in Theorem 5 are restricted to be sequences of *positive real numbers*, then the situation there is much simpler and requires no use of Césàro limits to handle. The hypothesis expressed in terms of equations (5) of this theorem need not be assumed to hold for all positive integers, but only for infinitely many (or just for arbitrarily large real numbers). The reader is asked to verify that this weaker assumption leads to the same conclusion. It then follows easily that if $A$ and $B$ are positive compact operators with kernels of the same dimension and if equation (8) has arbitrarily large real solutions $\lambda$, then $A$ and $B$ are unitarily equivalent. The same conclusion holds if equation (8) has any strictly increasing sequence of real solutions, even if bounded, for then the analyticity of both sides of the equation as functions of the complex variable $\lambda$ implies equality throughout their common domain that includes arbitrarily large real $\lambda$. $\square$

Niven and Zuckerman [4] discuss density of sequences of integers.

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