

COMPACT WEIGHTED COMPOSITION OPERATORS ON L^p

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ABSTRACT. We characterize the compact weighted composition operators on L^p ($1 \leq p < \infty$) and determine their spectra. We also show that every weakly compact weighted composition operator on L^1 is compact.

Let B be a Banach space consisting of (equivalence classes of) complex functions on a set X with the pointwise operations. A weighted composition operator on B is a bounded linear operator T from B into B that has the form: $Tf(x) = u(x)f(\varphi(x))$ for all $x \in X$ and $f \in B$, where φ is a self-map of X and u is a function on X . This operator T is often denoted by uC_φ . In [3] W. Feldman takes B to be a Banach F -lattice, roughly speaking, a certain Banach space with a lattice structure, and has shown that a weighted composition operator uC_φ on B satisfying some conditions is compact if and only if

$$(1) \quad \varphi(\{x \in X : |u(x)| \geq \varepsilon\}) \text{ is finite for each } \varepsilon > 0.$$

However, condition (1) is not necessary for the compactness of a weighted composition operator uC_φ on the L^p -spaces. Indeed, there exists a compact operator uC_φ on L^p that does not satisfy (1). Such operators may be found in [8, Theorem 2.3, Examples 2.1, 2.2; 9, Example 3.5]. In this note, we focus on the case that B is an L^p -space.

Let (X, \mathcal{M}, μ) be a σ -finite measure space. We abbreviate the Lebesgue space $L^p(X, \mathcal{M}, \mu)$ to L^p , and denote the L^p -norm by $\|\cdot\|_p$ ($1 \leq p \leq \infty$). Take $u \in L^\infty$, and set $S(u) = \{x \in X : u(x) \neq 0\}$. Let φ be a measurable transformation from $S(u)$ into X , and suppose that the measure μ_φ , defined by $\mu_\varphi(E) = \mu(\varphi^{-1}(E))$ for all $E \in \mathcal{M}$, is absolutely continuous with respect to μ (we write $\mu_\varphi \ll \mu$, as usual). Then we can define the linear operator uC_φ from L^p into the space of equivalence classes of measurable functions as

$$uC_\varphi f(x) = \begin{cases} u(x)f(\varphi(x)) & \text{if } x \in S(u), \\ 0 & \text{if } x \in X \setminus S(u), \end{cases}$$

for all $f \in L^p$. We use the assumption $\mu_\varphi \ll \mu$ to see that uC_φ is well defined as a mapping of equivalence classes of functions. If uC_φ is a bounded linear

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operator from L^p into L^p , then we say that uC_φ is a *weighted composition operator on L^p* . In the case that u is the constant function 1, the corresponding operator uC_φ on L^p is said to be a *composition operator*.

Compact (weighted) composition operators on L^2 have been studied by Singh, Kumar, and Dharmadhikari [7–10], and the related various topics including their research were reported in [5]. Along the line of their arguments, we deal with the compact weighted composition operators on L^p . In the sequel, we fix p with $1 \leq p < \infty$, unless otherwise qualified.

1. PRELIMINARIES

For above u and φ , we define the measure $\mu_{u^p, \varphi}$ by

$$\mu_{u^p, \varphi}(E) = \int_{\varphi^{-1}(E)} |u|^p d\mu, \quad E \in \mathcal{M}.$$

Then $\mu_{u^p, \varphi}$ is absolutely continuous with respect to μ , because the assumption $\mu_\varphi \ll \mu$ implies that for each $E \in \mathcal{M}$ with $\mu(E) = 0$, $\mu(\varphi^{-1}(E)) = 0$, and so $\mu_{u^p, \varphi}(E) = \int_{\varphi^{-1}(E)} |u|^p d\mu = 0$. Consequently, there exists the Radon-Nikodým derivative $d\mu_{u^p, \varphi}/d\mu$, which, of course, is a nonnegative L^1 -function. Put $\theta = (d\mu_{u^p, \varphi}/d\mu)^{1/p}$. We next consider the following operator M_θ from L^p into the space of equivalence classes of measurable functions

$$M_\theta f = \theta \cdot f, \quad f \in L^p.$$

The operator M_θ is closely related to uC_φ by the quantity

$$(2) \quad \|uC_\varphi f\|_p = \|M_\theta f\|_p, \quad f \in L^p,$$

which is obtained through the computation

$$\begin{aligned} \int |uC_\varphi f|^p d\mu &= \int |u|^p |f \circ \varphi|^p d\mu = \int |f|^p d\mu_{u^p, \varphi} \\ &= \int |f|^p \frac{d\mu_{u^p, \varphi}}{d\mu} d\mu = \int |f|^p \theta^p d\mu = \int |M_\theta f|^p d\mu. \end{aligned}$$

From equality (2), we see that the boundedness of uC_φ is equivalent to the boundedness of M_θ . Moreover, it is well known that M_θ is a *multiplication operator on L^p* , that is, a bounded linear operator on L^p into L^p , if and only if θ is essentially bounded. Also, equality (2) shows the implication (a) \Leftrightarrow (b) in the following lemma:

Lemma 1. *Let uC_φ be a weighted composition operator on L^p and M_θ the associated multiplication operator. Then the following conditions are equivalent.*

- (a) uC_φ is compact.
- (b) M_θ is compact.
- (c) For any $\varepsilon > 0$, the restriction of L^p to the set $\{x \in X : \theta(x) \geq \varepsilon\}$ is finite dimensional.

To prove the remainder, (b) \Leftrightarrow (c), we refer to Lemma 1.1 of [8], which is, with the same proof, true for the general setting of L^p .

2. COMPACTNESS

As is well known, a σ -finite measure space (X, \mathcal{M}, μ) is decomposed into two disjoint sets Y and Z , where Y does not possess any atoms and Z is a

countable union of atoms of finite measure. Moreover, we can easily check the following facts.

(a) Every L^p -function is constant μ -almost everywhere on any atom in Z ($1 \leq p \leq \infty$).

(b) If φ is as in our setting, φ maps an atom in $Z \cap S(u)$ essentially into an atom in Z , that is, for any atom A with $\mu(A \cap S(u)) > 0$, there exists an atom A' such that $\mu(A \cap \varphi^{-1}(A')) = \mu(A)$.

Thanks to these facts, we may assume without loss of generality that each atom in Z is a point mass. So we write $Z = \{z_1, z_2, \dots\}$ and put $a_n = \mu(\{z_n\}) > 0$. We characterize the compact weighted composition operators on L^p as follows.

Theorem 1. *Let uC_φ be a weighted composition operator on L^p . Then uC_φ is compact if and only if φ maps $S(u)$ essentially into Z , that is,*

$$(3) \quad \mu(\varphi^{-1}(Y)) = 0$$

and

$$(4) \quad \lim_{n \rightarrow \infty} \frac{1}{a_n} \int_{\varphi^{-1}(\{z_n\})} |u|^p d\mu = 0,$$

where we understand the limit takes the value zero when Z is a finite union of atoms.

Proof. Let $Y_0 = \{y \in Y : \theta(y) > 0\}$. We first observe that the condition (3) is equivalent to $\mu(Y_0) = 0$. Notice that

$$\int_Y \theta^p d\mu = \int_Y \frac{d\mu_{u^p, \varphi}}{d\mu} d\mu = \mu_{u^p, \varphi}(Y) = \int_{\varphi^{-1}(Y)} |u|^p d\mu.$$

This equality gives the following information: θ is zero μ -almost everywhere on Y if and only if u is zero μ -almost everywhere on $\varphi^{-1}(Y)$. In other words, $\mu(Y_0) = 0$ is equivalent to $\mu(\varphi^{-1}(Y)) = \mu(\varphi^{-1}(Y) \cap S(u)) = 0$. We next note that (4) means $\lim_{n \rightarrow \infty} \theta(z_n) = 0$. It results from a few simple equalities,

$$a_n \theta^p(z_n) = \int_{\{z_n\}} \theta^p d\mu = \int_{\{z_n\}} \frac{d\mu_{u^p, \varphi}}{d\mu} d\mu = \mu_{u^p, \varphi}(\{z_n\}) = \int_{\varphi^{-1}(\{z_n\})} |u|^p d\mu,$$

whence, $\theta(z_n) = (\frac{1}{a_n} \int_{\varphi^{-1}(\{z_n\})} |u|^p d\mu)^{1/p}$. Thus we need only show that the assertion (c) in Lemma 1 is equivalent to $\mu(Y_0) = 0$ and $\lim_{n \rightarrow \infty} \theta(z_n) = 0$.

For each $\varepsilon > 0$, let $X_\varepsilon = \{x \in X : \theta(x) \geq \varepsilon\}$ and $L_\varepsilon^p = L^p|_{X_\varepsilon}$, the restriction of L^p to X_ε . If $\mu(Y_0) = 0$ and $\lim_{n \rightarrow \infty} \theta(z_n) = 0$, for each $\varepsilon > 0$, X_ε is the union of finitely many atoms and a set of μ -measure zero, so L_ε^p is finite dimensional.

Conversely, suppose that either $\mu(Y_0) = 0$ or $\lim_{n \rightarrow \infty} \theta(z_n) = 0$ is false. In the case that $\mu(Y_0) \neq 0$, we can take an $\varepsilon > 0$ such that the set $Y_\varepsilon = \{y \in Y : \theta(y) \geq \varepsilon\}$ has positive μ -measure. Since Y_ε does not contain any atoms, there exists a decreasing sequence $\{E_n\}$ of subsets of Y_ε with $0 < \mu(E_n) < 1/n$. Then the sequence $\{\chi_{E_n}\}$ of the characteristic functions of E_n spans an infinite-dimensional subspace of L_ε^p , and therefore L_ε^p is infinite dimensional. On the other hand, if $\lim_{n \rightarrow \infty} \theta(z_n) \neq 0$, the set $\{z \in Z : \theta(z) \geq \varepsilon\}$ contains infinitely many atoms for some $\varepsilon > 0$, which implies that L_ε^p is infinite dimensional. Thus the theorem is proved. \square

Let us consider the special case of Theorem 1, where μ is nonatomic. When $X = Y$, the quantity (3) implies that $\mu(S(u)) = \mu(\varphi^{-1}(X)) = \mu(\varphi^{-1}(Y)) = 0$. It means that u is zero μ -almost everywhere on X , that is, uC_φ is a zero operator. Thus we have

Corollary. *Suppose that μ is nonatomic. Then a weighted composition operator on $L^p(X, \mathcal{M}, \mu)$ is compact if and only if it is a zero operator. In particular, no composition operator on $L^p(X, \mathcal{M}, \mu)$ is compact.*

Similar results had been proved in [7, Theorem 1; 9, Corollary 3.2; 10, Theorem 3.6]. Our corollary is a generalization of those results.

3. SPECTRA

Our next task is about the spectra. For composition operators on L^p and weighted ones on l^2 , some properties of their spectra were described by Ridge [6] and Carlson [1]. In this section, we determine the spectrum $\sigma(uC_\varphi)$ of a compact weighted composition operator uC_φ on L^p .

We begin with the definition of the k th iterate φ_k of the map $\varphi : S(u) \rightarrow X$. Set $W_0 = X$, and put $\varphi_0(x) = x$ for all $x \in W_0$. Inductively define W_k and φ_k as $W_k = \{x \in W_{k-1} : \varphi_{k-1}(x) \in S(u)\}$ and $\varphi_k(x) = \varphi(\varphi_{k-1}(x))$, for all $x \in W_k$. Clearly the domains W_k of φ_k form a decreasing sequence of X . An atom z is called a *fixed atom* of φ of order k ($k \geq 1$) if z is in W_k , and if $\varphi_k(z) = z$ and $\varphi_j(z) \neq z$, $j = 1, \dots, k-1$. From (b) in the first paragraph in §2, we know that $\{z, \varphi(z), \dots, \varphi_k(z)\} \subset Z$ for any atom z in $Z \cap W_k$.

Theorem 2. *Let uC_φ be a compact weighted composition operator on L^p . If we set $\Lambda = \{\lambda : \lambda^k = u(z)u(\varphi(z)) \cdots u(\varphi_{k-1}(z))\}$, for some fixed atom z of order k , then we have*

$$\sigma(uC_\varphi) \cup \{0\} = \Lambda \cup \{0\}.$$

Proof. To prove the theorem, we adopt the method by Kamowitz [4]. Actually, the proof of $\sigma(uC_\varphi) \cup \{0\} \supset \Lambda \cup \{0\}$ is the same as for Proposition 3 in [4]. Here we show the opposite inclusion: $\sigma(uC_\varphi) \cup \{0\} \subset \Lambda \cup \{0\}$.

Let $\lambda \notin \Lambda \cup \{0\}$, and suppose that an L^p -function f satisfies $\lambda f = uC_\varphi f$. All that we have to show is that f is zero μ -almost everywhere on X . For, if this holds, λ is not an eigenvalue of uC_φ , and by the Fredholm alternative for compact operators, λ is not in $\sigma(uC_\varphi)$, and thus we obtain $\sigma(uC_\varphi) \cup \{0\} \subset \Lambda \cup \{0\}$. We break its proof into two steps.

(i) The first step is to show that f is zero on Z . Pick $z \in Z$. Since $X = \bigcup_{k=0}^{\infty} W_k$, z is in W_k for some k . By induction, we can easily show that $\lambda^k f(z) = u(z)u(\varphi(z)) \cdots u(\varphi_{k-1}(z))f(\varphi_k(z))$. We use this equality frequently.

Here we add the assumption that z is not in W_{k+1} . Then $\varphi_k(z) \notin S(u)$ yields $\lambda f(\varphi_k(z)) = uC_\varphi f(\varphi_k(z)) = 0$. Using the above equality, we have $\lambda^{k+1} f(z) = u(z) \cdots u(\varphi_{k-1}(z))\lambda f(\varphi_k(z)) = u(z) \cdots u(\varphi_{k-1}(z))uC_\varphi f(\varphi_k(z)) = 0$. Since $\lambda \neq 0$, we get $f(z) = 0$.

From now on, we may assume that z is in W_k for any k . We first consider the case that $\mathcal{O}(z) = \{z, \varphi(z), \varphi_2(z), \dots\}$ is finite. Then, for some j , $\varphi_j(z)$ is a fixed atom of φ . If k is the order of this fixed atom $\varphi_j(z) = z'$, it is easy to see that $\lambda^k f(z') = u(z') \cdots u(\varphi_{k-1}(z'))f(z')$. Since the hypothesis $\lambda \notin \Lambda$ says that $\lambda^k \neq u(z') \cdots u(\varphi_{k-1}(z'))$, we obtain $f(z') = 0$, i.e., $f(\varphi_j(z)) = 0$.

Furthermore, we have $\lambda^j f(z) = u(z) \cdots u(\varphi_{j-1}(z))f(\varphi_j(z)) = 0$, hence $f(z) = 0$.

We next consider the remaining case, that $\mathcal{O}(z)$ is infinite. Let us assume that

$$\left\{ k : \sqrt[p]{\frac{\mu(\{\varphi_k(z)\})}{\mu(\{\varphi_{k+1}(z)\})}} |u(\varphi_k(z))| \geq \varepsilon \right\}$$

is infinite for some $\varepsilon > 0$. Since

$$\sqrt[p]{\frac{\mu(\{\varphi_k(z)\})}{\mu(\{\varphi_{k+1}(z)\})}} |u(\varphi_k(z))| \geq \varepsilon$$

yields

$$\left(\frac{1}{\mu(\{\varphi_{k+1}(z)\})} \int_{\varphi^{-1}(\{\varphi_{k+1}(z)\})} |u|^p d\mu \right)^{1/p} \geq \varepsilon,$$

and the elements of $\mathcal{O}(z)$ are distinct, we have infinitely many n satisfying $((a_n)^{-1} \int_{\varphi^{-1}(\{z_n\})} |u|^p d\mu)^{1/p} \geq \varepsilon$. This is contrary to condition (4) in Theorem 1. Hence, for any $\varepsilon > 0$, there exists a K such that $\sqrt[p]{\mu(\{\varphi_k(z)\})/\mu(\{\varphi_{k+1}(z)\})} \times |u(\varphi_k(z))| < \varepsilon$ for all $k \geq K$. Then, for each $k \geq K$, we have

$$\begin{aligned} |\lambda^k f(z)| &= |u(z) \cdots u(\varphi_{K-1}(z))u(\varphi_K(z)) \cdots u(\varphi_{k-1}(z))f(\varphi_k(z))| \\ &= |u(z)| \cdots |u(\varphi_{K-1}(z))| \frac{1}{\sqrt[p]{\mu(\{\varphi_K(z)\})}} \sqrt[p]{\frac{\mu(\{\varphi_K(z)\})}{\mu(\{\varphi_{K+1}(z)\})}} |u(\varphi_K(z))| \\ &\quad \cdots \sqrt[p]{\frac{\mu(\{\varphi_{k-1}(z)\})}{\mu(\{\varphi_k(z)\})}} |u(\varphi_{k-1}(z))| \sqrt[p]{|f(\varphi_k(z))|^p \mu(\{\varphi_k(z)\})} \\ &< \|u\|_\infty^K \frac{1}{\sqrt[p]{\mu(\{\varphi_K(z)\})}} \varepsilon^{k-K} \left(\int |f|^p d\mu \right)^{1/p} = \frac{\|u\|_\infty^K \|f\|_p}{\varepsilon^K \sqrt[p]{\mu(\{\varphi_K(z)\})}} \varepsilon^k, \end{aligned}$$

so

$$|f(z)| < \frac{\|u\|_\infty^K \|f\|_p}{\varepsilon^K \sqrt[p]{\mu(\{\varphi_K(z)\})}} \left(\frac{\varepsilon}{|\lambda|} \right)^k.$$

As $\varepsilon = |\lambda|/2$ and $k \rightarrow \infty$, this inequality forces $f(z) = 0$. Thus we conclude that f is zero on Z .

(ii) In this step, we show that f is zero μ -almost everywhere on Y . Decompose Y into three parts: $Y \setminus S(u)$, $Y \cap \varphi^{-1}(Z)$, and $Y \cap \varphi^{-1}(Y)$. If $y \in Y \setminus S(u)$, then $\lambda f(y) = u C_\varphi f(y) = 0$, and so $f(y) = 0$ because $\lambda \neq 0$. Suppose that $y \in Y \cap \varphi^{-1}(Z)$. Since $\varphi(y) \in Z$, the preceding step leads to $f(\varphi(y)) = 0$. Hence $\lambda f(y) = u C_\varphi f(y) = u(y)f(\varphi(y)) = 0$, so $f(y) = 0$. Thus we have shown that f is zero on $(Y \setminus S(u)) \cup (Y \cap \varphi^{-1}(Z))$, while Theorem 1(3) tells us that the remaining part $Y \cap \varphi^{-1}(Y)$ has μ -measure zero, completing the second step.

Evidently, steps (i) and (ii) show that f is zero μ -almost everywhere on X . \square

4. WEAK COMPACTNESS

Finally we investigate the weakly compact weighted composition operators on L^p . For $1 < p < \infty$, L^p is reflexive, so every weighted composition operator on L^p is weakly compact. But when $p = 1$, the situation is different.

Theorem 3. *A weighted composition operator on L^1 is weakly compact if and only if it is compact.*

For the proof, we shall need the next lemma, which can be quickly deduced from Theorem IV.8.9, and its Corollaries 8.10, 8.11 in [2].

Lemma 2. *Let H be a weakly sequentially compact set in L^1 . For each decreasing sequence $\{E_n\}$ in \mathcal{M} such that $\lim_{n \rightarrow \infty} \mu(E_n) = 0$ or $\bigcap_{n=1}^{\infty} E_n = \emptyset$, the sequence of integrals $\{\int_{E_n} |h| d\mu\}$ converges to zero uniformly for h in H .*

Proof of Theorem 3. It suffices to show the “only if” part. Let uC_φ be a weakly compact weighted composition operator on L^1 . To see that uC_φ is compact, we prove conditions (3) and (4) in Theorem 1.

We first show that for each $\varepsilon > 0$, the set $Y_\varepsilon = \{y \in Y : \theta(y) \geq \varepsilon\}$ has μ -measure zero. To this end, assume that $\mu(Y_\varepsilon) > 0$ for some $\varepsilon > 0$. Using the fact that Y_ε has no atom, we find a decreasing sequence $\{F_n\}$ of subsets of Y_ε with $0 < \mu(F_n) < \frac{1}{n}$. Then we have $\mu(\bigcap_{n=1}^{\infty} F_n) = 0$, and the assumption $\mu_\varphi \ll \mu$ gives $\mu(\varphi^{-1}(\bigcap_{n=1}^{\infty} F_n)) = 0$. Consider the decreasing sequence $\{\varphi^{-1}(F_n)\}$. Since the Radon-Nikodým derivative $d\mu_\varphi/d\mu$ is a nonnegative L^1 -function, it follows that $\mu(\varphi^{-1}(F_1)) = \mu_\varphi(F_1) = \int_{F_1} (d\mu_\varphi/d\mu) d\mu < \infty$. With the aid of this finiteness, we see that

$$\lim_{n \rightarrow \infty} \mu(\varphi^{-1}(F_n)) = \mu\left(\bigcap_{n=1}^{\infty} \varphi^{-1}(F_n)\right) = \mu\left(\varphi^{-1}\left(\bigcap_{n=1}^{\infty} F_n\right)\right) = 0,$$

that is, $\{\varphi^{-1}(F_n)\}$ satisfies the first condition of the decreasing sequence in Lemma 2. Now put $L_B^1 = \{f \in L^1 : \|f\|_1 \leq 1\}$. Since $uC_\varphi L_B^1$ is weakly sequentially compact, we can apply Lemma 2, with $H = uC_\varphi L_B^1$ and $E_n = \varphi^{-1}(F_n)$. As a consequence, there exists an N such that

$$(5) \quad \int_{\varphi^{-1}(F_N)} |uC_\varphi f| d\mu < \varepsilon, \quad f \in L_B^1.$$

On the other hand, for the characteristic function χ_{F_N} of F_N , we have

$$\begin{aligned} \int_{\varphi^{-1}(F_N)} \left| uC_\varphi \left(\frac{1}{\mu(E_N)} \chi_{F_N} \right) \right| d\mu &= \frac{1}{\mu(F_N)} \int_{\varphi^{-1}(F_N)} |u| |\chi_{F_N} \circ \varphi| d\mu \\ &= \frac{1}{\mu(F_N)} \int_{\varphi^{-1}(F_N)} |u| d\mu = \frac{1}{\mu(F_N)} \mu_{u^1, \varphi}(F_N) = \frac{1}{\mu(F_N)} \int_{F_N} \theta d\mu \geq \varepsilon, \end{aligned}$$

because $\theta \geq \varepsilon$ on Y_ε , in particular on F_N . Since $\chi_{F_N}/\mu(F_N)$ is in L_B^1 , this contradicts (5). Hence $\mu(Y_\varepsilon) = 0$ for any $\varepsilon > 0$, and therefore, $\mu(Y_0) = 0$. As we saw in the proof of Theorem 1, this is equivalent to (3), which is one condition to be proved.

We next show condition (4). This time we define the decreasing sequence $\{F_n\}$ as $F_n = \{z_k : k \geq n\}$. Then the decreasing sequence $\{\varphi^{-1}(F_n)\}$ satisfies the second condition of Lemma 2, namely, $\bigcap_{n=1}^{\infty} \varphi^{-1}(F_n) = \emptyset$. Applying Lemma 2 once more, we conclude that (5) holds for some F_N in this $\{F_n\}$. For any n with $n \geq N$, let f_n be the function that has a value 1 at z_n and vanishes elsewhere. Since f_n/a_n is in L_B^1 , taking $f = f_n/a_n$ in (5), we have

$$\varepsilon > \int_{\varphi^{-1}(F_N)} \left| u C_{\varphi} \left(\frac{1}{a_n} f_n \right) \right| d\mu = \frac{1}{a_n} \int_{\varphi^{-1}(\{z_n\})} |u| d\mu.$$

Since this holds for any $n \geq N$, we obtain (4), which is the other required condition to complete the proof. \square

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