NONTRIVIAL SOLUTIONS
OF SEMILINEAR ELLIPTIC EQUATIONS
WITH CONTINUOUS OR DISCONTINUOUS NONLINEARITIES

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Abstract. In this paper, we are concerned with the boundary value problem of the form $-\Delta u = g(u)$ in $\Omega$, $u|_{\partial \Omega} = 0$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, under assumptions of relations between $g$ and the eigenvalues of $-\Delta$. If $g$ is piecewise continuous on any bounded closed interval in $\mathbb{R}$, the above equation takes the form $-\Delta u \in [g(u), \tilde{g}(u)]$ in $\Omega$, $u|_{\partial \Omega} = 0$. We obtain the existence of nontrivial solutions in both resonant and nonresonant cases at 0. Our theorems can be also applied when $g$ is discontinuous (may be discontinuous at 0).

1. Introduction

We begin this paper by considering the existence of nontrivial solutions of the boundary value problem of the form

$$-\Delta u = g(u) \quad \text{in} \quad \Omega,$$

where $\Omega$ is a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^n$ and $g$ is a real-valued continuous function on $\mathbb{R}$ such that $g(0) = 0$.

Let $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ denote the eigenvalues of the selfadjoint realization in $L^2(\Omega)$ of $-\Delta$ with the Dirichlet boundary condition. Many authors have studied the existence of nontrivial solutions of the problem (1) when $g(t)/t$ crosses finitely many eigenvalues of $-\Delta$ as $t$ varies from $-\infty$ to $+\infty$. Amann and Zehnder [2] proved by generalized Morse theory that (1) has at least one nontrivial solution if $g \in C^2(\mathbb{R}, \mathbb{R})$ satisfies $\sup_{t \in \mathbb{R}} |g'(t)| < \infty$ and

$$\lambda_{k-1} < g'(0) < \lambda_k \leq \lambda_m < a_* \leq a^* < \lambda_{m+1}$$

for some $m, k \geq 1$

where

$$a_* = \liminf_{|t| \to \infty} \frac{g(t)}{t} \quad \text{and} \quad a^* = \limsup_{|t| \to \infty} \frac{g(t)}{t}.$$
On the other hand, using the Leray-Schauder degree theory, Hirano [6] established the existence of one nontrivial solution of (1) under

\[ \lambda_{k-1} < b_* < b^* < \lambda_k \leq \lambda_m < a_* \leq a^* < \lambda_{m+1} \quad \text{for some } k, m \geq 1, \]

where \( a_* \) and \( a^* \) are as above, and

\[ b_* = \liminf_{|t| \to 0} \frac{g(t)}{t} \quad \text{and} \quad b^* = \limsup_{|t| \to 0} \frac{g(t)}{t}, \]

without any assumptions of differentiability of \( g \). Hirano's result cannot be applied in the case of resonance at 0. We obtain the existence of one nontrivial solution of (1) under weaker conditions of \( g \) near 0 that contain the resonance case at 0 (Theorem 1). Moreover, there are no results for \( g \) with \( b_* > a^* \) in [6]. We deal with such a function \( g \) in Theorem 2.

It is seen in §3 that the assertions of Theorems 1 and 2 remain valid in the case that \( g \) is a piecewise continuous function on any bounded closed interval of \( \mathbb{R} \) (may be discontinuous at 0), that is,

\[ -\Delta u \in [g(u), \bar{g}(u)] \quad \text{in } \Omega \]

\[ u|_{\partial \Omega} = 0, \]

where

\[ \overline{g}(t) = \liminf_{s \to t} g(s) \quad \text{and} \quad \underline{g}(t) = \limsup_{s \to t} g(s). \]

2. THE CASE THAT \( g \) IS CONTINUOUS

Our purpose in this section is to prove the following two theorems.

**Theorem 1.** Let \( g: \mathbb{R} \to \mathbb{R} \) be a continuous function with \( g(0) = 0 \). If \( g \) satisfies the condition

\[ b^* < \lambda_m < a_* \leq \overline{a} < \lambda_{m+1} \]

for some \( m \geq 1 \), where

\[ a_0 = \liminf_{|t| \to \infty} \frac{g(t)}{t}, \quad \overline{a} = \sup_{t \neq 0} \frac{g(t)}{t} \quad \text{and} \quad b^* = \limsup_{|t| \to 0} \frac{g(t)}{t}, \]

then equation (1) has at least one nontrivial solution in \( H^2(\Omega) \cap H_0^1(\Omega) \).

**Theorem 2.** Let \( g: \mathbb{R} \to \mathbb{R} \) be a continuous function with \( g(0) = 0 \). If \( g \) satisfies

\[ \lambda_{k-1} < a \leq a^* < \lambda_k < b_* \]

for some \( k \geq 1 \), where

\[ a = \inf_{t \neq 0} \frac{g(t)}{t}, \quad a^* = \limsup_{|t| \to \infty} \frac{g(t)}{t} \quad \text{and} \quad b_* = \liminf_{|t| \to 0} \frac{g(t)}{t}, \]

then there exists at least one nontrivial solution of (1) in \( H^2(\Omega) \cap H_0^1(\Omega) \).

In the following, we write \( H, H^{-1} \), and \( L^2 \) instead of \( H_0^1(\Omega), H^{-1}(\Omega), \) and \( L^2(\Omega) \), respectively. We denote by \( \| \cdot \|, \| \cdot \|_* \), and \( | \cdot | \) the norms of \( H, H^{-1} \), and \( L^2 \), respectively. The notation \( | \cdot | \) is often used for the absolute value of a real number without notice if there is no possibility of
their confusion. The pairing between $H$ and $H^{-1}$ is denoted by $\langle \cdot , \cdot \rangle$. We take $k \in \mathbb{Z}^+$ with $b^* < \lambda_k \leq \lambda_m$ if $g$ satisfies condition (3), and $m \in \mathbb{Z}^+$ with $\lambda_k \leq \lambda_m < b^*$ if $g$ satisfies condition (4). Let $H_1, H_2$, and $H_3$ be closed subspaces of $H$ spanned by the eigenfunctions corresponding to the eigenvalues \( \{ \lambda_{m+1}, \lambda_{m+2}, \ldots \} \), \( \{ \lambda_k, \ldots, \lambda_m \} \), and \( \{ \lambda_1, \lambda_2, \ldots, \lambda_{k-1} \} \), respectively. (We consider $\lambda_0 = 0$ and $H_3 = \{0\}$ if $k = 1$.)

For $i = 1, 2, 3$, $P_i$ means the projection from $H$ onto $H_i$. Define a real-valued function $f$ on $H$ by

\[
(5) \quad f(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_0^{u(x)} g(t) \, dt \, dx \quad \text{for} \ u \in H.
\]

Then we have

\[
\langle f'(u), v \rangle = \langle -\Delta u - g(u), v \rangle \quad \text{for any} \ u, v \in H,
\]

and hence weak solutions of (1) coincide with critical points of $f$.

We need the following two lemmas in order to prove our theorems.

**Lemma 1.** If $g$ satisfies condition (3) or (4), then the Palais-Smale condition holds for the function $f$ defined by (5), that is, for any sequence $\{u_n\}$ in $H$ such that $\{f(u_n)\}$ is bounded and $\|f'(u_n)\|_* \to 0$ there exists a convergent subsequence of $\{u_n\}$.

**Proof.** Let $\{u_n\}$ in $H$ satisfy that $\{f(u_n)\}$ is bounded and $\|f'(u_n)\|_* = \| - \Delta u - g(u)\|_* \to 0$. For each $u_n$, we put $v_n = P_1 u_n$, $w_n = P_2 u_n$ and $z_n = P_3 u_n$. Then

\[
\langle -\Delta u_n - g(u_n), v_n - (w_n + z_n) \rangle = \|v_n\|^2 - \|w_n + z_n\|^2 - \int_H g(u_n)(v_n - (w_n + z_n)) \, dx.
\]

Suppose that $g$ satisfies condition (3). Then there exist positive numbers $\alpha$ with $\lambda_m < \alpha < a_*$ and $\rho$ such that $\alpha \leq g(t)/t \leq \overline{a}$ for all $t \in \mathbb{R}$ with $|t| \geq \rho$. From the continuity of $g$, for some constant $K$, we have $|g(t)| \leq K$ for all $t$ with $|t| < \rho$. If $|u_n(x)| \geq \rho$ then

\[
(6) \quad \alpha \leq \frac{g(u_n(x))}{v_n(x) + w_n(x) + z_n(x)} \leq \overline{a}.
\]

If $|u_n(x)| < \rho$ then

\[
|v_n(x)|^2 - |w_n(x) + z_n(x)|^2 \geq -\rho(|v_n(x)| + |w_n(x) + z_n(x)|).
\]

We set

\[
A = \{x \in \Omega: |v_n(x)| > |w_n(x) + z_n(x)|\},
\]

\[
A_1 = \{x \in A: |u_n(x)| \geq \rho\}, \quad A_2 = \{x \in A: |u_n(x)| < \rho\}.
\]

By the second inequality in (6), we have

\[
\int_A g(u_n)(v_n - (w_n + z_n)) \, dx
\]

\[
\leq \int_{A_1} \overline{a}(|v_n|^2 - |w_n + z_n|^2) \, dx + \int_{A_2} K(|v_n| + |w_n + z_n|) \, dx
\]

\[
\leq \int_A (\overline{a}|v_n|^2 - \alpha|w_n + z_n|^2) \, dx + \int_{A_2} (\overline{a}\rho + K)(|v_n| + |w_n + z_n|) \, dx.
\]

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Putting

\[ B = \{ x \in \Omega : |v_n(x)| \leq |w_n(x) + z_n(x)| \}, \]

\[ B_1 = \{ x \in B : |u_n(x)| \geq \rho \}, \quad B_2 = \{ x \in B : |u_n(x)| < \rho \}, \]

it follows that

\[ \int_B g(u_n)(v_n - (w_n + z_n)) \, dx \]

\[ \leq \int B (\bar{a}|v_n|^2 - \alpha|w_n + z_n|^2) \, dx + \int_{B_2} (\bar{a}\rho + K)(|v_n| + |w_n + z_n|) \, dx \]

from the first inequality in (6). Therefore we have

\[ \int_{\Omega} g(u_n)(v_n - (w_n + z_n)) \, dx \]

\[ \leq \bar{a}|v_n|^2 - \alpha|w_n + z_n|^2 + 2|\Omega|^{1/2}(\bar{a}\rho + K)|u_n|. \]

Thus it holds that

\[ \langle -\Delta u_n - g(u_n), v_n - (w_n + z_n) \rangle \]

\[ \geq \left( 1 - \frac{\bar{a}}{\lambda_{m+1}} \right) \|v_n\|^2 + \left( \frac{\alpha}{\lambda_m} - 1 \right) \|w_n + z_n\|^2 - 2|\Omega|^{1/2}(\bar{a}\rho + K)|u_n| \]

for some \( \omega_1, \omega_2 > 0 \). The assumption \( \| -\Delta u_n - g(u_n) \|_* \to 0 \) and this inequality imply the boundedness of \( \{ u_n \} \) in \( H \) and hence the existence of a subsequence \( \{ u_{n_j} \} \) of \( \{ u_n \} \) that converges weakly to some \( u \) in \( H \). Then we have

\[ \langle -\Delta u_{n_j} - g(u_{n_j}), u_{n_j} - u \rangle \to 0. \]

Since \( H \) is compactly embedded into \( L^2 \), \( \{ u_{n_j} \} \) strongly converges to \( u \) in \( L^2 \) and \( \langle g(u_{n_j}), u_{n_j} - u \rangle \to 0 \), so \( \langle -\Delta u_{n_j}, u_{n_j} - u \rangle \to 0 \). Since \( \{-\Delta u_n\} \) weakly converges to \( -\Delta u \) in \( H^{-1} \), we have

\[ \lim_{j \to \infty} \|u_{n_j}\|^2 = \lim_{j \to \infty} \langle -\Delta u_{n_j}, u_{n_j} - u \rangle + \lim_{j \to \infty} \langle -\Delta u_{n_j}, u \rangle = \|u\|^2. \]

Thus we obtain the strong convergence of \( \{ u_{n_j} \} \) in \( H \). The proof is similar in the case that \( g \) satisfies condition (4).

**Lemma 2.** Under assumption (3), there exist positive constants \( c_i (i = 1, 2, 3, 4), \epsilon_j (j = 1, 2), \) and \( K \) such that

(i) if \( \|P_1u\| \geq c_1, \|P_2u\| \leq c_2, \) and \( \|P_3u\| \leq c_3, \) then \( f(u) \geq \epsilon_1; \)

(ii) if \( \|P_2u\| \leq c_4 \) and \( \|P_3u\| \leq K\|P_2u\|, \) then \( f(u) \geq \epsilon_2\|P_2u\|^2. \)

**Proof.** For simplicity, we set \( v = P_1u, \ w = P_2u, \) and \( z = P_3u. \) By \( \bar{a} < \lambda_{m+1}, \) we have

\[ f(u) \geq \frac{1}{2}\|v + w + z\|^2 - \frac{1}{2}\bar{a}\|v + w + z\|^2 \]

\[ \geq \frac{1}{2} \left\{ \left( 1 - \frac{\bar{a}}{\lambda_{m+1}} \right) \|v\|^2 - \left( \frac{\bar{a}}{\lambda_k} - 1 \right) \|w\|^2 - \left( \frac{\bar{a}}{\lambda_k - 1} \right) \|z\|^2 \right\}, \]

so there exist positive constants \( c_i (i = 1, 2, 3) \) and \( \epsilon_1 \) for which (i) holds. From \( b^* < \lambda_k, \) we obtain positive constants \( \delta \) and \( \alpha \) with \( \alpha < \lambda_k \) such that
$g(t)/t \leq \alpha$ for all $t$ with $|t| \leq \delta$. In the case that $|v(x) + w(x) + z(x)| \leq \delta$, we have
\[
\frac{1}{2}(\lambda_{m+1}|v|^2 + \lambda_k|w|^2 + \lambda_1|z|^2) - \int_0^{v+w+z} g(t) \, dt
\geq \frac{1}{2}(\lambda_{m+1} - \alpha)|v|^2 + \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \alpha)|z|^2 - \alpha(vw + wz + zv)
\geq \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \alpha)|z|^2 - \alpha(vw + wz + zv).
\]
Now, we choose $d > 0$ such that
\[
(\lambda_{m+1} - \alpha)p^2 + 2(\lambda_{m+1} - \alpha)pq + (\bar{a} - \alpha)q^2 \leq (\lambda_{m+1} - \bar{a})\delta^2
\]
for all $p, q \geq 0$ with $p + q \leq d$. Moreover, we can take $c > 0$ such that
\[
\sup_{x \in \Omega}(|P_2u(x)| + |P_3u(x)|) \leq d
\]
if $\|P_2u + P_3u\| \leq c$. Let $\|w + z\| \leq c$. In the case that $|v(x) + w(x) + z(x)| > \delta$, we have
\[
\left|\int_0^{v+w+z} g(t) \, dt\right| \leq \frac{1}{2}\bar{a}(v + w + z)^2 - \frac{1}{2}(\bar{a} - \alpha)\delta^2
\]
and hence
\[
\frac{1}{2}(\lambda_{m+1}|v|^2 + \lambda_k|w|^2 + \lambda_1|z|^2) - \int_0^{v+w+z} g(t) \, dt
\geq \frac{1}{2}(\lambda_{m+1} - \bar{a})\left\{\left|v + \frac{\alpha - \bar{a}}{\lambda_{m+1} - \bar{a}}(|w| + |z|)\right|^2
\right.
- \frac{\bar{a} - \alpha}{2(\lambda_{m+1} - \bar{a})}\{(\lambda_{m+1} - \alpha)|w|^2 + 2(\lambda_{m+1} - \alpha)|w||z|
\left. + (\bar{a} - \alpha)|z|^2 - (\lambda_{m+1} - \bar{a})\delta^2\}
\]
\[
+ \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \bar{a})|z|^2 - \alpha(vw + wz + zv)
\geq \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \bar{a})|z|^2 - \alpha(vw + wz + zv).
\]
It follows that
\[
f(u) \geq \int_\Omega \left\{\frac{1}{2}(\lambda_{m+1}|v|^2 + \lambda_k|w|^2 + \lambda_1|z|^2) - \int_0^{v+w+z} g(t) \, dt\right\} \, dx
\]
\[
\geq \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \bar{a})|z|^2 \geq \frac{1}{2}\left\{\frac{\lambda_k - \alpha}{\lambda_m}\|w\|^2 - \frac{\bar{a} - \lambda_1}{\lambda_1}\|z\|^2\right\}
\]
if $\|w + z\| \leq c$. Taking $K, c_4, \text{and } c_2$ such that
\[
0 < K < \sqrt{\frac{\lambda_1(\lambda_k - \alpha)}{\lambda_m(\bar{a} - \lambda_1)}}, \quad 0 < (1 + K)c_4 \leq c,
\]
and
\[
0 < c_2 < \frac{\lambda_k - \alpha}{2\lambda_m} \left(1 - K^2\frac{\lambda_m(\bar{a} - \lambda_1)}{\lambda_1(\lambda_k - \alpha)}\right),
\]
(ii) holds.
We are now ready to prove Theorem 1.

**Proof of Theorem 1.** By \( \lambda_m < a_* \leq \overline{a} < \lambda_{m+1} \), there exists \( r > 0 \) such that
\( f(w + z) < \inf_{v \in H} f(v) \) for all \( w \in H_2 \) and \( z \in H_3 \) with \( \|w + z\| \geq r \). We define
\[
\Gamma^* = \{ A \subset H : A \text{ is a compact set such that } \sigma(A) \ni 0 \}
\]
for any continuous mapping \( \sigma : A \to H_2 \oplus H_3 \) satisfying \( \sigma(u) = u \) for all \( u \in A \cap S \) \( (\neq \emptyset) \),

where
\[
S = \{ w + z : w \in H_2 , \ z \in H_3 , \text{ and } \|w + z\| = r \}
\]
and
\[
c^* = \inf_{A \in \Gamma^*} \max_A \left( \geq \inf_{v \in H} f(v) \right).
\]

It is easily seen that if \( A \in \Gamma^* \) and \( \eta : A \to H \) is a continuous mapping such that \( \eta(u) = u \) for all \( u \in A \cap S \), then \( \eta(A) \in \Gamma^* \). Since \( f \) satisfies the Palais-Smale condition by Lemma 1, \( c^* \) is a critical value of \( f \) by a method similar to Rabinowitz's saddle point theorem \([9, 7]\). Assume that 0 is the only critical point of \( f \). Let \( c_i \ (i = 1, 2, 3, 4) \), \( \epsilon_j \ (j = 1, 2) \), and \( K \) be positive numbers in Lemma 2. We set
\[
U = \{ u \in H : \|P_1 u\| < a , \ \|P_2 u\| < b , \ \text{and} \ \|P_3 u\| < c/2 \}
\]
and
\[
V = \{ u \in H : \|P_1 u\| < a , \ \|P_2 u\| < b , \ \text{and} \ \|P_3 u\| < c \},
\]
where \( a = c_1 , \ b = \min\{c_2 , c_4 \} , \ \text{and} \ c = \min\{c_3 , K b \} \). We may suppose that \( r > \sqrt{b^2 + c^2} \) with no loss of generality. Putting \( \gamma = \min\{\epsilon_1 , \epsilon_2 b^2 \} \), it follows that \( f \geq \gamma \) on \( \{ u \in H : \|P_1 u\| \leq a , \ \|P_2 u\| \leq b , \ \text{and} \ \|P_3 u\| \leq c \} \cup \{ u \in H : P_1 u \| \leq a , \ \|P_2 u\| = b , \ \text{and} \ \|P_3 u\| \leq c \} \). From \( c^* = 0 \), for \( 0 < \epsilon < \gamma \), there exists \( A \in \Gamma^* \) with \( \max_A f < \epsilon \). Now, we define \( T : H \to H \) by
\[
T(u) = \begin{cases} u & \text{if } u \notin V, \\ \varphi(\|P_3 u\|)(P_1 + P_2)u + P_3 u & \text{if } u \in V, \end{cases}
\]
where \( \varphi : [0, +\infty) \to [0, 1] \) is defined by
\[
\varphi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq c/2, \\ (2/c)t - 1 & \text{if } c/2 < t \leq c, \\ 1 & \text{if } c < t. \end{cases}
\]

Then, \( T \) is continuous on \( \{ u \in H : \|P_1 u\| = a , \ \|P_2 u\| \leq b , \ \text{and} \ \|P_3 u\| \leq c \} \cap \{ u \in H : \|P_1 u\| \leq a , \ \|P_2 u\| = b , \ \text{and} \ \|P_3 u\| \leq c \} \). By \( \dim H_2 \neq 0 \), we can choose \( w_0 \in H_2 \) with \( 0 < \|w_0\| < b/2 \). Define \( \tilde{T} : T(A) \to H \) by
\[
\tilde{T}(u) = \begin{cases} u & \text{if } \|P_3 u\| \geq c/2, \\ P_1 u + Q((P_2 + P_3)u) & \text{if } \|P_3 u\| < c/2, \end{cases}
\]
where \( Q((P_2 + P_3)u) \) means the intersection of the half line \( \{ t(P_2 + P_3)u + (1-t)w_0 : t \geq 0 \} \) and the relative boundary of \( \{ w + z : w \in H_2 , \ z \in H_3 , \ \|w\| < b , \ \text{and} \ \|z\| < c/2 \} \) in \( H_2 \oplus H_3 \). Putting \( \sigma = (P_2 + P_3) \circ \tilde{T} \circ T \), \( \sigma \) is a
continuous mapping from $A$ into $H_2 \oplus H_3$ such that $\sigma(u) = u$ for all $u \in A \cap S$. Since $f \geq \gamma > \varepsilon$ on $\{u \in H : \|P_1 u\| \geq a, \|P_2 u\| \leq b, \text{ and } \|P_3 u\| \leq c\}$, we have $\sigma(A) \not\equiv 0$. This is contrary to $A \in \Gamma^*$. This completes the proof.

Next we prove Theorem 2.

**Proof of Theorem 2.** From $\lambda_{k-1} < \mu < \mu^* < \lambda_k$, we take $r > 0$ largely enough such that $f(z) < \inf_{\nu \in H_1, \, \omega \in H_2} f(\nu + \omega)$ for all $z \in H_3$ with $\|z\| \geq r$. We set $B = \{z \in H_3 : \|z\| \leq r\}$ and $S = \{z \in H_3 : \|z\| = r\}$. Define

$$
\Gamma = \{g : g \text{ is a continuous mapping from } B \text{ into } H \text{ such that } g(z) = z \text{ for all } z \in S\} \quad (\neq \emptyset)
$$

and

$$
c = \inf_{g \in \Gamma} \sup_{z \in B} f(g(z)) = \inf_{\nu \in H_1, \, \omega \in H_2} \inf_{z \in B} f(\nu + \omega).$$

Similarly to the proof of Theorem 1, $c$ is a critical value of $f$. Now, suppose that $f$ does not have any nonzero critical points in $H$. From $\mu > \lambda_{k-1}$, it follows that

$$f(z) < \frac{1}{2} \|z\|^2 - \frac{1}{2} \lambda_{k-1} |z|^2 \leq 0 \quad \text{for all } z \in H_3.$$ 

By $b^* > \lambda_m$, there exists $\delta > 0$ such that $g(t)/t > \lambda_m$ for all $t$ with $|t| \leq \delta$. Then, we obtain $c_1 > 0$ such that $\sup_{x \in \Omega} |w(x) + z(x)| \leq \delta$ if $w \in H_2$, $z \in H_3$, and $\|w + z\| \leq c_1$. Therefore we have

$$f(w + z) \leq \frac{1}{2} \|w + z\|^2 - \frac{1}{2} \lambda_m |w + z|^2 \leq 0$$

for all $w \in H_2$ and $z \in H_3$ with $\|w + z\| \leq c_1$. We may assume $c_1 < r$ without loss of generality. Choosing $c_2 > 0$ arbitrarily we put

$$U = \{u \in H : \|P_1 u\| < c_2 \text{ and } \|P_2 + P_3 u\| < c_1/2\}.$$ 

Since $\dim H_2 \neq 0$, by an argument similar to the proof of Theorem 1, we can construct a continuous mapping $g : B \to H$ such that $g(z) = z$ for all $z \in S$, $g(B) \cap U = \emptyset$, and $f(g(z)) \leq 0$ for all $z \in B$. From the well-known deformation lemma, for sufficiently small $\varepsilon_0 > 0$, there exist a continuous mapping $\eta : H \to H$ and a positive number $\varepsilon < \varepsilon_0$ satisfying the conditions

(i) $\eta(u) = u$ if $u \notin f^{-1}([-\varepsilon_0, \varepsilon_0])$;
(ii) $\eta(f^{-1}(\langle -\infty, \varepsilon \rangle) \setminus U) \subset f^{-1}(\langle -\infty, -\varepsilon \rangle) \setminus \{0\}$.

Putting $\tilde{g} = \eta \circ g$, it is clear that $\tilde{g} \in \Gamma$. On the other hand, $\max_{z \in B} f(\tilde{g}(z)) \leq -\varepsilon$ since $g(B) \cap U = \emptyset$. This is contrary to $c = 0$. This completes the proof.

3. The case that $g$ is discontinuous

In this section, we consider the existence of one nontrivial solution of equation (2). Let $g : R \to R$ be a piecewise continuous function on any bounded closed interval (may be discontinuous at 0) with $0 \in [g(0), \overline{g}(0)]$. Then, it is easily seen that the functional $f$ defined by (5) is locally Lipschitz continuous if $g$ satisfies condition (3) or (4). Then, we cannot apply the usual critical point theory for differentiable functionals since $f$ may be nondifferentiable. In order to solve the problem (2), Chang [4] made use of the generalized gradients
for locally Lipschitz continuous functionals introduced by Clarke [5]. In fact, it was shown that
\[ \partial f(u) \subset -\Delta u - [g(u), \bar{g}(u)] \quad \text{for each } u \in H, \]
where \( \partial f(u) \) means the generalized gradient of \( f \) at \( u \).

Further, he proved in [4] that the deformation lemma holds in this case. On the other hand, Mizoguchi [8] obtained the existence of one nontrivial solution of (2) under the same conditions as Theorem 1 in [6].

We remark that \( g \) is automatically continuous at 0 in [8]. According to the proofs of Theorems 1 and 2, we see that equation (2) has at least one nontrivial solution if the condition (3) or (4) is assumed.

**Theorem 3.** Let \( g: \mathbb{R} \to \mathbb{R} \) be a piecewise continuous function on any bounded closed interval with \( 0 \in [g(0), \bar{g}(0)] \). If \( g \) satisfies condition (3), then equation (2) has at least one nontrivial solution in \( H^2(\Omega) \cap H^1_0(\Omega) \).

**Theorem 4.** Let \( g: \mathbb{R} \to \mathbb{R} \) be a piecewise continuous function on any bounded closed interval with \( 0 \in [g(0), \bar{g}(0)] \). If \( g \) satisfies condition (4), then there exists at least one nontrivial solution of (2) in \( H^2(\Omega) \cap H^1_0(\Omega) \).

**References**


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