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REPRESENTING KNOT GROUPS INTO SL(2, C)

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ABSTRACT. We show that if a knot in S^3 has nontrivial Alexander polynomial then the fundamental group of its complement has a representation into $\mathrm{SL}(2, \mathbf{C})$ whose image contains a free group of rank two.

Since the advent of Casson's invariant, one of the intriguing aspects of representations of the fundamental groups of three-dimensional $\mathbf{Z}$-homology spheres is the question of whether the group of every such homology sphere other than S^3 has an irreducible representation into $\mathrm{SU}(2)$. This is also of considerable relevance to Floer-Donaldsen theory. The question in this generality seems to be some way from resolution (one corollary is the Poincaré conjecture!). However we may retreat to a weaker question by observing that if this were true then every homology sphere obtainable by surgery on a nontrivial knot in S^3 has an irreducible representation into $\mathrm{SL}(2, \mathbf{C})$ and in particular, an affirmative answer implies that every nonabelian classical knot group has such a representation. This latter question also arises naturally in the context of [CS, CC].

In this note we prove the following

Theorem 1. *Let K be a knot in S^3 whose Alexander polynomial is not identically 1. Then $G(K) = \pi_1(S^3 \backslash K)$ has a representation into $\mathrm{SL}(2, \mathbf{C})$ whose image contains a free group of rank two.*

The reason for specifying that the image contain a free group rather than only requiring that it be nonsoluble is that we wish to rule out the rather uninteresting case of dihedral representations that although irreducible are guaranteed by the condition that $\Delta_K(-1) \neq 1$. Such representations are elliptic and already exist in $\mathrm{SU}(2)$.

We also remark that the condition on the Alexander polynomial guarantees that $G(K)$ has a representation into $\mathrm{SL}(2, \mathbf{C})$ that is soluble and nonabelian, (see [BZ] or [R]) and a natural way to try and prove Theorem 1 would be to show that any such representation can be deformed to an irreducible one. However our proof does not proceed in this way and it remains an interesting open question whether a knot in S^3 can have all its soluble representations being isolated.

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One interpretation of Theorem 1 is that the Alexander polynomial is a strictly weaker invariant of knottedness than the A-polynomial of [CC].

Corollary 2. *If K is a knot in S^3 with nontrivial Alexander polynomial, then it has nontrivial A-polynomial.*

Various special cases are also of interest, for example we have

Corollary 3. *The group of every fibered knot in S^3 has a representation into* $\mathrm{SL}(2, \mathbf{C})$ *whose image contains a free group of rank two.*

Corollary 2 follows immediately from Theorem 1, since the latter guarantees a representation that by [T], has to lie on a component of representations of complex dimension at least 1, so that there is at least one factor in the A-polynomial of K.

Proof of Theorem 1. We note that a calculation from the presentation (see [BZ]) reveals that the theorem is true for torus knots and it is certainly true for hyperbolic knots.

It follows from Thurston's hyperbolisation theorem that we can suppose K contains a nonboundary parallel incompressible embedded torus, so that K is a satellite knot. By Alexander's theorem, every torus embedded into S^3 bounds a solid torus on at least one side and it follows that K contains an incompressible torus T that separates the knot complement into two pieces, one of which is an atoriodal knot complement in S^3 and the other of which is a solid torus containing the knot K. Observe that since the exterior knot is atoriodal, it has a representation of the required sort.

As usual we define the algebraic (resp. geometric) winding numbers of K inside the solid torus to be the minimal number of algebraic (resp. geometric) intersections of K with a meridian disc of the solid torus. Note that the geometric winding is strictly larger than zero since T is incompressible.

There are now two cases. Suppose that the algebraic winding number of K is nonzero. Then we see easily that we may extend a random irreducible representation of the exterior knot across to K by using an abelian representation of $\pi_1(D^2 \times S^1 \backslash K)$; the representation of $G(K)$ so obtained satisfies the conclusions of the theorem provided the representation of the exterior knot did.

We are therefore reduced to the case that the algebraic winding number of K inside the solid torus is zero. Mark T using the longitude-meridian pair coming from the exterior knot. We may re-embed into S^3 so that it is unknotted and so that the longitude on T now bounds a disc in S^3. The embedding yields a new knot, which we denote K_1. Notice that the winding number condition ensures that K has a Seifert surface lying entirely inside $D^2 \times S^1$ and our choice of embedding implies that the Seifert form of K_1 using this Seifert surface is the same as that of K. Whence K_1 also has the same Alexander polynomial as K, and in particular K_1 is knotted.

Further, there is a degree one map from the complement of K to the complement of K_1 so that there is a surjective homomorphism $G(K) \to G(K_1)$, so that if we can prove the theorem for the knot K_1, we will be done.

We may repeat the above argument to obtain a chain of knots K, K_1, $K_2, \ldots$ all of which have the same Alexander polynomial, with the property that either K_n has a representation of the required sort and hence so does K, or we may form K_{n+1}. We conclude the proof then by a result of [S] where it is shown that a chain of the above type cannot extend indefinitely without eventually terminating in the unknot. Since the Alexander polynomial condition excludes this last possibility, we must eventually reach a knot group for which we can find a representation and we are done.

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