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NIELSEN NUMBERS OF PERIODIC MAPS ON SOLVMANIFOLDS

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ABSTRACT. Let $f: M \to M$ be a self-map of a solvmanifold M. Then the Lefschetz number $L(f)$ and the Nielsen number $N(f)$ of f satisfy $|L(f)| \leq N(f)$. If f is homotopically periodic (i.e., $f^k \simeq \mathrm{id}$, for some $k \geq 1$), then $L(f) = N(f)$.

Let M be a closed manifold and let $f: M \to M$ be a continuous map. We study the relations between the two invariants; the *Lefschetz number* $L(f)$ and the *Nielsen number* $N(f)$.

These two numbers give information on the existence of fixed point sets. If $L(f) \neq 0$, every self-map g of M homotopic to f has a nonempty fixed point set. The Nielsen number is a lower bound for the number of components of the fixed point set of all maps homotopic to f.

Even though $N(f)$ gives more information than $L(f)$ does, it is harder to calculate. For some cases, it is known that they are almost the same.

If M is a torus, then $|L(f)| = N(f)$ [2].

If M is a nilmanifild, then $|L(f)| = N(f)$ [1].

If M is an infranilmanifold and f is homotopically periodic, then $L(f) = N(f)$ [6]. In fact, $|L(f)| = N(f)$ cannot be generalized to infranilmanifolds or solvmanifolds as [6, Theorem 10] shows. In [6] it was also shown that $L(f) = N(f)$ cannot be generalized to nonperiodic maps. In this paper, we show $L(f) = N(f)$ holds for solvmanifolds M if f has finite homotopy period.

Let S be a connected, simply connected solvable Lie group and H be a closed subgroup of S. The coset space $H\backslash S$ is called a solvmanifold. We shall talk about *compact* solvmanifolds only.

Theorem. *Let $f: M \to M$ be a self-map of a solvmanifold M. Then $|L(f)| \leq N(f)$. If f is homotopically periodic then $L(f) = N(f)$.*

The first part of the statement, which generalizes [1], has been proved in [9]. The second part is a generalization of certain cases of [6]. Note that there are

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infranilmanifolds that are not solvmanifolds. The proof uses the same idea as in [9], but many things had to be clarified; especially, the existence of the fully invariant subgroup and the construction of a model space M' are explicitly shown. At the final stage, we use the method of [1, 6] to get the result.

For a map $f: M \to M$, define an equivalence relation on $\mathrm{Fix}(f)$ as follows: For $x_0, x_1 \in \mathrm{Fix}(f)$, $x_0 \sim x_1$ if and only if there exists a path c from x_0 to x_1 such that c is homotopic to $f \circ c$ relative to the end points. An equivalence class of this relation is called a *fixed point class* (= FPC) of f. Let $p: \widetilde{M} \to M$ be a universal covering of M. It is known that $x_0 \sim x_1$ if and only if $x_0, x_1 \in p\,\mathrm{Fix}(\tilde{f})$ for some lift $\tilde{f}: \widetilde{M} \to \widetilde{M}$ of f to the universal covering. Therefore, a fixed point class is of the form $p\,\mathrm{Fix}(\tilde{f})$ for some lift $\tilde{f}: \widetilde{M} \to \widetilde{M}$. For each fixed point class C of f, there is an integer associated with it, called the index of the FPC. Then the fixed point class C is called *essential* if the $\mathrm{index}(f, C)$ is nonzero. See [5, Theorem 1.6].

The rest of this paper is a proof of the theorem. First we construct a new model space M' that is homotopy equivalent to M and a map $f': M' \to M'$ corresponding to f. The space M' is a fiber bundle over a torus with a fiber nilmanifold; and f' will be fiber-preserving. To use the local product formula for the index of fixed point components, we apply the technique and result of [6, Theorem 2].

Let $\pi = \pi_1(M)$ be the fundamental group of $M = H\backslash S$, where S is a connected, simply connected solvable Lie group and H is a closed subgroup. Then $\pi = H/H_0$. Such a group π is known to be a *strongly torsionfree* $\mathcal{S}$ *group*; that is, π contains a finitely generated, torsionfree nilpotent normal subgroup D with the quotient π/D free abelian of finite rank.

Consider the commutator subgroup $[\pi, \pi]$. Let F be the maximal finite subgroup of the quotient group $\pi/[\pi, \pi]$. Then F is fully invariant (i.e., F is invariant under any endomorphism of $\pi/[\pi, \pi]$). Let Γ be the inverse image of F under the quotient map $\pi \to \pi/[\pi, \pi]$. Then since $[\pi, \pi]$ is fully invariant in π, Γ is a fully invariant subgroup of π. We show that Γ is nilpotent. Since $\pi/D \cong \mathbb{Z}^s$ is torsionfree, the torsion subgroup of $\pi/[\pi, \pi]$ lies in $D/[\pi, \pi]$. This implies that $\Gamma \subset D$, and hence Γ is nilpotent. We have found a fully invariant subgroup Γ of π that is nilpotent and $\pi/\Gamma \cong \mathbb{Z}^s$ is free abelian.

With the exact sequence of groups

$$1 \to \Gamma \to \pi \to \mathbb{Z}^s \to 1$$

we shall do the Seifert fiber space construction to obtain a bundle over a torus with a nilmanifold as a fiber. It is very important that the fundamental group of the fiber is fully invariant in π because the maps involved are not just automorphisms but endomorphisms.

By Mal′cev, there is a unique connected, simply connected nilpotent Lie group G containing Γ as a lattice. Let $C(\mathbb{R}^s, G^*)$ be the group of all smooth maps of $\mathbb{R}^s$ into G. The group law is

$$(\lambda * \eta)(w) = \eta(w) \cdot \lambda(w)$$

for $\lambda, \eta \in C(\mathbb{R}^s, G^*)$ and $w \in \mathbb{R}^s$. Let $E(s) = \mathbb{R}^s \rtimes O(s)$ be the Euclidean group acting on $\mathbb{R}^s$. By choosing a coordinate system on $\mathbb{R}^s$, $E(s)$ is the group of isometries of $\mathbb{R}^s$. We can embed our $\mathbb{Z}^s$ as standard translations via

$\rho: \mathbb{Z}^s \to \mathbb{R}^s \subset E(s)$. The group $\mathrm{Aut}(G) \times E(s)$ acts on $C(\mathbb{R}^s, G^*)$ by

$$(g, h) \cdot \lambda = g \circ \lambda \circ h^{-1}$$

for $(g, h) \in \mathrm{Aut}(G) \times E(s)$ and $\lambda \in C(\mathbb{R}^s, G^*)$. Now form the semidirect product

$$\mathscr{U} = C(\mathbb{R}^s, G^*) \rtimes (\mathrm{Aut}(G) \times E(s)).$$

The group law is

$$(\lambda, g, h)(\lambda_1, g_1, h_1) = (\lambda * g\lambda_1 h^{-1}, gg_1, hh_1),$$

and it acts on $G \times \mathbb{R}^s$ by

$$(\lambda, g, h)(x, w) = (g(x) \cdot \lambda(h(w)), h(w)).$$

From the short exact sequence $1 \to \Gamma \to \pi \to \mathbb{Z}^s \to 1$ together with the action of $\mathbb{Z}^s$ on $\mathbb{R}^s$, we construct a representation of π into $\mathscr{U} = C(\mathbb{R}^s, G^*) \rtimes (\mathrm{Aut}(G) \times E(s))$. The lattice $\Gamma \subset G$ sits in $C(\mathbb{R}^s, G^*) \rtimes \mathrm{Inn}(G)$ as left multiplications $(a, \mu(a))$, where $\mu(a)(x) = axa^{-1}$.

In [7], it is proved that, in this situation, there exists a representation $\pi \to \mathscr{U}$ so that the following diagram with exact rows is commutative.

$$\begin{array}{ccccccccc}
1 & \to & \Gamma & \to & \pi & \to & \mathbb{Z}^s & \to & 1 \\
 & & \downarrow & & \downarrow \theta & & \downarrow \psi\times\rho & & \\
1 & \to & C(\mathbb{R}^s, G^*) \rtimes \mathrm{Inn}(G) & \to & \mathscr{U} & \to & \mathrm{Out}(G) \times E(s) & \to & 1
\end{array}$$

Furthermore, such a θ is unique up to conjugation by elements of $C(\mathbb{R}^s, G^*) \rtimes \mathrm{Inn}(G)$. These are based on the fact that $H^i(\mathbb{Z}^s; C(\mathbb{R}^s, G^*)) = 0$ for $i = 1, 2$. These are nonabelian group cohomologies.

The action of π on $G \times \mathbb{R}^s$ by this representation is properly discontinuous and free because π is torsionfree. Thus, the representation of π into $\mathscr{U}$ gives rise to a compact smooth manifold $M' = \pi\backslash(G \times \mathbb{R}^s)$. The manifold M' has a (Seifert) fiber structure

$$\Gamma\backslash G \to M' \to \mathbb{R}^s/\mathbb{Z}^s = T^s$$

with the fiber nilmanifold $N = \Gamma\backslash G$.

Note that π is a torsionfree virtually poly $\mathbb{Z}$-group. According to Farrell-Hsiang, M and M' are homeomorphic if $\dim M \neq 3, 4$. However, we do not need this fact. We only need the fact that M and M' are homotopy equivalent. Let $\alpha: M \to M'$ be a homotopy equivalence with a homotopy inverse β. Let $f': M' \to M'$ be the composite $\alpha \circ f \circ \beta$. Then the homotopy commutative diagram

$$\begin{array}{ccc}
M & \xleftarrow{\beta} & M' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
M & \xrightarrow{\alpha} & M'
\end{array}$$

implies that $N(f) = N(f')$ by the homotopy invariance of the Nielsen number. See [5, Theorem I.5.4]. Of course, $L(f) = L(f')$ since the Lefschetz number is a homology invariant.

Thus we shall prove the theorem for M' and $f'\colon M' \to M'$. For convenience, from now on, we denote M' and f' by just M and f. Therefore our M is the total space of the bundle $\Gamma\backslash G \to M \to T^s$. Since Γ is fully invariant, $f_\#\colon \pi_1 M \to \pi_1 M$ maps Γ into itself.

We argue that there exists a fiber-preserving map $M \to M$ that is homotopic to f. Let $(F, e) \xrightarrow{i} (E, e) \xrightarrow{p} (B, b)$ be a fibration of connected pointed spaces. Assume B is an aspherical space. Let $f\colon E \to E$ be a map preserving the base point. Assume that the induced homomorphism $f_\#\colon \pi_1(E, e) \to \pi_1(E, e)$ leaves the subgroup $(f \circ i)_\#(\pi_1(F, e))$ invariant. When f did not preserve the base point, one can homotope f so that $f(e) = e$. Note that the homotoping does not change the invariance of the subgroup $f_\#\colon \pi_1(E, e) \to \pi_1(E, e)$. Then $f_\#$ induces a homomorphism $\phi\colon \pi_1(B, b) \to \pi_1(B, b)$ so that $p_\# \circ f_\# = \phi \circ p_\#$. Since B is aspherical, there is a map $g\colon (B, b) \to (B, b)$ inducing ϕ. Then $p \circ f \simeq g \circ p$. By the Covering Homotopy Theorem, there exists $H\colon E \times I \to E$ covering the homotopy $p \circ f \simeq g \circ p$ and $H(x, 0) = f(x)$ for all $x \in E$. Now define $f'\colon E \to E$ by $f'(x) = H(x, 1)$. Since $p(H(x, 1)) = g(p(x))$, f' is fiber-preserving, and is homotopic to f. Note that the spaces F and E need not be aspherical spaces in the argument.

Going back to our model space M, we assume that f is fiber-preserving. In fact, when $f_\#$ has finite period, such a map f can be found so that it has a lift in the group $\mathscr{U}$.

Let $\bar{f}\colon T \to T$ be the induced map on the base space by f. If $L(\bar{f}) = 0$, then $\bar{f}$ is homotopic to a fixed point free map $\bar{f}'$. Using the homotopy lifting property of the fibration, we can homotope f to a fixed point free map f'. Thus, $L(f) = 0 = N(f)$.

From now on, we assume $L(\bar{f}) \neq 0$. Let $\bar{f}_*$ be the endomorphism of $\pi_1(T^s)$ induced by $\bar{f}$. Consider $\bar{f}$ as an element of $\mathrm{gl}(s, \mathbb{Z})$. By [2], $\det(I - \bar{f}_*) = L(\bar{f})$, which is nonzero by assumption. This implies that $\bar{f}$ is homotopic to a map with finitely many isolated fixed points. (Observe that the solution of the equation $g(x) = x$ on $\mathbb{R}^s$ for an affine map $g(x) = Ax + a$, has only one solution if $\det(I - A) \neq 0$). So let $\mathrm{Fix}(\bar{f}) = \{b_1, b_2, \dots, b_m\}$. Moreover, all these fixed points have the same index, sign(det($I - A$)), which is either $+1$ or -1. Let $\tilde{f}\colon \widetilde{M} \to \widetilde{M}$ be a lift of f to the universal covering and $\tilde{f}_{\tilde{b}}$ be the restriction of $\tilde{f}$ to $\widetilde{N}_{\tilde{b}}$, the fiber of $\widetilde{M} \to \widetilde{T}$ over $\tilde{b} \in \widetilde{T}$. Note that $\widetilde{N}_{\tilde{b}}$ is a universal cover of N_b. We shall apply

Theorem [5, IV 3.4, p. 86]. *An* FPC *of f is essential if and only if its projection in T is an essential* FPC *of $\bar{f}$ and its intersection with an invariant fiber consists of essential fixed point classes of f_b.*

Note that all fixed points $b_1, b_2, \dots, b_m$ are essential fixed point classes of $\bar{f}$. For each FPC $p\,\mathrm{Fix}(\tilde{f})$ of f, there exists $b \in \mathrm{Fix}(\bar{f})$ such that $p\mathrm{Fix}(\tilde{f}) = p\,\mathrm{Fix}(\tilde{f}_{\tilde{b}}) \subset N_{b_i}$. Moreover, by the above theorem, $p\mathrm{Fix}(\tilde{f})$ is an essential FPC of f if and only if $p\,\mathrm{Fix}(\tilde{f}_{\tilde{b}})$ is an essential FPC of f_b. Therefore,

$$N(f) = \#(\text{essential FPC of } f) = \sum_i \#(\text{essential FPC of } f \text{ over } b_i)$$

$$= \sum_i \#(\text{essential FPC of } f_{b_i}) = \sum_i N(f_{b_i}).$$

On the other hand, we have

$$\begin{aligned} L(f) &= \operatorname{index}\left(f, \bigcup_i \operatorname{Fix}(f_{b_i})\right) = \sum_i \operatorname{index}(f, \operatorname{Fix}(f_{b_i})) \\ &= \sum_i \operatorname{index}(\overline{f}, b_i) \cdot \operatorname{index}(f_{b_i}, \operatorname{Fix}(f_{b_i})) = \sum_i \operatorname{index}(\overline{f}, b_i) \cdot L(f_{b_i}). \end{aligned}$$

See [5, IV.3.1, p. 84]. Since $\operatorname{index}(\overline{f}, b_i) = +1$ or -1 and is constant for all i,

$$|L(f)| = \left|\sum \pm L(f_{b_i})\right| = \left|\sum L(f_{b_i})\right| \le \sum |L(f_{b_i})|.$$

However, it is known that $|L(f_{b_i})| = N(f_{b_i})$ on a nilmanifold N_{b_i} by [1] or [4]. Thus, $|L(f)| \le \sum_i |L(f_{b_i})| = \sum N(f_{b_i}) = N(f)$.

Now suppose that f has homotopically finite period. By [6], $L(\overline{f}) = N(\overline{f})$. This implies that $\operatorname{index}(\overline{f}, b_i) = +1$ for all i, and hence $L(f) = \sum_i L(f_{b_i})$. Now f_{b_i} has homotopically finite period. Again by [6], $L(f_{b_i}) = N(f_{b_i})$ for each i. Consequently,

$$L(f) = \sum_i L(f_{b_i}) = \sum N(f_{b_i}) = N(f).$$

This completes the proof of theorem. □

Remarks. To generalize the theorem to infrasolvmanifolds, it seems that one needs to have a product formula for Seifert fiber spaces. Such spaces have singularities, and we do not have a product formula for generalized bundles.
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