A NOTE ON THE COMMUTANTS OF CSL ALGEBRAS MODULO BIMODULES

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(Communicated by Paul S. Muhly)

Abstract. In this note, we show that for any σ-weakly closed bimodule M of a CSL algebra A satisfying M ⊇ A, the commutant of A modulo M is equal to M itself. Theorem 6 provided a result on the cohomology groups of CSL subalgebras of Von Neumann algebras.

Throughout this paper, L(H) will be the set of all bounded linear operators on a complex Hilbert space H. If $\mathcal{L}$ is a projection lattice in L(H) that is complete and $0, I \in \mathcal{L}$, then we denote by $\text{alg} \mathcal{L}$, the set $\{ T \in L(H): P^\perp TP = 0 \text{ for all } P \in \mathcal{L} \}$. $\text{alg} \mathcal{L}$ is said to be a CSL algebra if $\mathcal{L}$ is commutative. By a bimodule M of a subalgebra A of L(H) in L(H), we mean a linear subspace M of L(H) such that $AM \subseteq M$ and $MA \subseteq M$.

Let A be a Banach algebra and M be a (two-sided) Banach A-module. Then the space $C^n(A, M)$ of n-cochains $(n = 1, 2, \ldots)$ consists of all bounded n-linear maps from the product of n-copies of A to M. The coboundary operator $\partial$ from $C^n(A, M)$ to $C^{n+1}(A, M)$ is defined by

$$(\partial \phi)(a_0, a_1, \ldots, a_n) = a_0 \phi(a_1, \ldots, a_n) + \sum_{j=1}^{n} (-1)^j \phi(a_0, \ldots, a_j, a_{j+1}a_j, \ldots, a_n)$$

where $\phi \in C^n(A, M)$. Let $C^0(A, M) = M$ and $\partial: C^0(A, M) \to C^1(A, M)$ be defined by $\partial h(a) = ah - ha$ $(a \in A, h \in M)$.

Let $Z^n(A, M) = \{ \sigma \in C^n(A, M): \partial \sigma = 0 \}$,

$B^n(A, M) = \{ p \in C^n(A, M): p = \partial \phi \text{ for some } \phi \in C^{n-1}(A, M) \}$;

then $Z^n(A, M) \supseteq B^n(A, M)$ and the quotient space $Z^n(A, M)/B^n(A, M)$ is called the n-dimensional cohomology group of A with coefficients in M (for details see [3]).

If A is a subalgebra of L(H) and M is a bimodule of A in L(H), then the commutant of A modulo M is denoted by $C(A, M)$, i.e., $C(A, M) = \text{alg} \mathcal{L}$.
{\mathcal T} \in \mathcal L(H): \mathcal T S - S T \in \mathcal M \text{ for any } S \text{ in } \mathcal A}. \) In [1] Christensen proved that if \( \mathcal A \) is a CSL algebra and \( \mathcal M \) is a \( \sigma \)-weakly closed subalgebra of \( \mathcal L(H) \) containing \( \mathcal A \), then \( H^1(\mathcal A, \mathcal A) = H^1(\mathcal A, \mathcal M) \); therefore, if \( H^1(\mathcal A, \mathcal A) = 0 \) then \( C(\mathcal A, \mathcal M) = \mathcal A + \mathcal M = \mathcal M \). Unfortunately there exists a CSL algebra \( \mathcal A \) such that \( H^1(\mathcal A, \mathcal A) \neq 0 \). Thus we may ask the following question: does \( C(\mathcal A, \mathcal M) = \mathcal M \) for all CSL algebras \( \mathcal A \) and all \( \sigma \)-weakly closed bimodules \( \mathcal M \) containing \( \mathcal A \)? In [2] it is pointed out that this is true for completely distributive CSL algebras. In this note we show that this is true for all CSL algebras. We also prove a theorem on the cohomology groups of CSL subalgebras of Von Neumann algebra.

The proof of the following lemma is similar to the proof of Theorem 4.1 in [4].

**Lemma 1.** Let \( \mathcal B \) be a Von Neumann algebra and \( \mathcal L \subseteq \mathcal B \) be a complete commutative projection lattice. Let \( \mathcal A = \text{alg}\mathcal L \cap \mathcal B \) and \( \mathcal C \) be the Von Neumann algebra generated by \( \mathcal L \). Suppose that \( \mathcal M \) is a \( \sigma \)-weakly closed bimodule of \( \mathcal A \) in \( \mathcal L(H) \). If \( \varphi \in C^n(\mathcal A, \mathcal M) \) is such that \( \partial \varphi \) is equal to zero whenever any of its arguments lies in \( \mathcal C \) then there is an element \( \sigma \in C^{n-1}(\mathcal A, \mathcal M) \) such that \( \varphi - \partial \sigma \) is equal to zero whenever any of its arguments lies in \( \mathcal C \).

**Lemma 2.** Let \( \mathcal A \) and \( \mathcal C \) be as in Lemma 1 and \( \varphi \in C^n(\mathcal A, \mathcal B) \). Suppose that \( \varphi \) and \( \partial \varphi \) vanish whenever one of their arguments lies in \( \mathcal C \). Then \( \varphi \in C^n(\mathcal A, \mathcal A) \).

**Proof.** By the assumption and since \( \mathcal L \subseteq \mathcal C \), it is easy to verify that \( \varphi \in C^n(\mathcal A, \text{alg}\mathcal L) \) (see the proof of Lemma 2.2 in [5]), thus \( \varphi \in C^n(\mathcal A, \mathcal A) \).

**Lemma 3.** Let \( \mathcal A \) and \( \mathcal B \) be as in Lemma 1. Then \( C(\mathcal A, \mathcal A) \cap \mathcal B = \mathcal A \).

**Proof.** If \( \mathcal T \in C(\mathcal A, \mathcal A) \cap \mathcal B \), then \( \mathcal T E - ET \in \mathcal A \subseteq \text{alg}\mathcal L \) for any \( \mathcal E \) in \( \mathcal L \); thus \( \mathcal E^{1}(\mathcal T E - ET)\mathcal E = \mathcal E^{1}TE = 0 \), which implies \( \mathcal T \in \text{alg}\mathcal L \) and therefore \( \mathcal T \in \mathcal A \).

**Theorem 4.** Let \( \mathcal A, \mathcal B, \) and \( \mathcal C \) be as in Lemma 1 and let \( \mathcal M \) be a \( \sigma \)-weakly closed bimodule of \( \mathcal A \) in \( \mathcal L(H) \).

1. If \( \mathcal B \supseteq \mathcal M \supseteq \mathcal A \) then \( C(\mathcal A, \mathcal M) \cap \mathcal B = \mathcal M \).
2. If \( \text{alg}\mathcal L \supseteq \mathcal M \supseteq \mathcal A \) then \( C(\mathcal A, \mathcal M) \cap \mathcal B = \mathcal A \).
3. If \( \mathcal M \supseteq \text{alg}\mathcal L \) then \( C(\mathcal A, \mathcal M) = \mathcal M \).

**Proof.** (1) Let \( \mathcal T \in C(\mathcal A, \mathcal M) \cap \mathcal B \). By Lemma 1, there exists \( \mathcal h \in \mathcal M \) such that \( \mathcal T - \mathcal h \in \mathcal C \). Hence \( \mathcal T - \mathcal h \in \text{alg}\mathcal L \cap \mathcal B = \mathcal A \) and therefore \( \mathcal T = \mathcal T - \mathcal h + \mathcal h \in \mathcal M \).

(2) and (3) are similar to (1).

**Corollary 5.** If \( \mathcal A = \text{alg}\mathcal L \) is a CSL algebra then \( C(\mathcal A, \mathcal M) = \mathcal M \) for any \( \sigma \)-weakly closed bimodule \( \mathcal M \) of \( \mathcal A \) containing \( \mathcal A \).

**Theorem 6.** Let \( \mathcal A, \mathcal B, \mathcal C, \) and \( \mathcal L \) be as in Lemma 1. Suppose that \( \mathcal M \) is a \( \sigma \)-weakly closed bimodule of \( \mathcal A \) in \( \mathcal L(H) \).

1. If \( \mathcal B \supseteq \mathcal M \supseteq \mathcal A \) then \( H^n(\mathcal A, \mathcal M) = H^n(\mathcal A, \mathcal A) \).
2. If \( \mathcal M \supseteq \text{alg}\mathcal L \) then \( H^n(\mathcal A, \mathcal M) = H^n(\mathcal A, \text{alg}\mathcal L) \).

**Proof.** We only prove (1). The proof of (2) is similar to that of (1), so we leave it to the reader.

Since \( B^n(\mathcal A, \mathcal A) \subseteq B^n(\mathcal A, \mathcal M) \) and \( Z^n(\mathcal A, \mathcal A) \subseteq Z^n(\mathcal A, \mathcal M) \), we define \( \pi \) to be the canonical map from \( H^n(\mathcal A, \mathcal A) \) to \( H^n(\mathcal A, \mathcal M) \).
Let $\sigma \in Z^n(A, M)$. Then $\partial \sigma = 0$ and, by Lemma 1, there exists $\rho \in C^{n-1}(A, M)$ such that $\sigma - \partial \rho$ vanishes whenever any of its arguments lies in $C$. Define $\sigma_1 = \sigma - \partial \rho$. Then $\partial \sigma_1$ and $\sigma_1$ vanish whenever any of their arguments lies in $C$. By Lemma 2, we know that $\sigma_1 \in Z^n(A, A)$. Thus $\pi$ is surjective.

In order to complete the proof, we need to show that $\pi$ is injective.

Let $\sigma \in Z^n(A, A)$ and $\sigma = \partial \varphi$, where $\varphi \in C^{n-1}(A, M)$. We will show that there is $\rho \in C^{n-1}(A, A)$ such that $\sigma = \partial \rho$, and thus $\pi$ is injective.

(i) If $n = 1$ then $\varphi \in C^0(A, M) = M$, and for any $a$ in $A$ we have $\sigma(a) = \partial \varphi(a) = a \varphi - \varphi a$. Since $E$ and $\sigma(E)$ are in $A$, for any $E$ in $L$ we have $0 = E^{\perp} \sigma(E)E = E^{\perp}(E\varphi - \varphi E)E = -E^{\perp} \varphi E$, which implies $\varphi \in \text{alg}L$; therefore, $\varphi \in \text{alg}L \cap M \subseteq \text{alg}L \cap B = A$.

(ii) Suppose that $n \geq 2$. Since $\partial \sigma = 0$, by Lemma 1, there exists $\varphi_1$ in $C^{n-1}(A, A)$ such that $\sigma - \partial \varphi_1$ vanishes whenever any of its arguments lies in $C$. Again by Lemma 1, for $\varphi - \varphi_1$ there is $\varphi_2 \in C^{n-2}(A, M)$ such that $\varphi - \varphi_1 - \partial \varphi_2$ vanishes whenever any of its arguments lies in $C$. Using Lemma 2, for $\varphi - \varphi_1 - \partial \varphi_2$ we have that $\varphi - \varphi_1 - \partial \varphi_2 \in C^{n-1}(A, A)$. Let $\rho = \varphi - \partial \varphi_2$. Then $\rho = (\varphi - \varphi_1 - \partial \varphi_2) + \varphi_1 \in C^{n-1}(A, A)$ and $\partial \rho = \partial \varphi = \sigma$.

**Corollary 7.** If $A = \text{alg}L$ is a CSL algebra, then for any $\sigma$-weakly closed bimodule $M$ of $A$ which contains $A$, we have $H^n(A, A) = H^n(A, M)$.

**References**


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