PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 3, November 1992

A NOTE ON THE COMMUTANTS OF CSL ALGEBRAS MODULO BIMODULES

HAN DEGUANG

(Communicated by Paul S. Muhly)

ABSTRACT. In this note, we show that for any σ-weakly closed bimodule M of a CSL algebra A satisfying $M \supseteq A$, the commutant of A modulo M is equal to M itself. Theorem 6 provided a result on the cohomology groups of CSL subalgebras of Von Neumann algebras.

Throughout this paper, $L(H)$ will be the set of all bounded linear operators on a complex Hilbert space H. if $\mathscr{L}$ is a projection lattice in $L(H)$ that is complete and $0, I \in \mathscr{L}$, then we denote by $\operatorname{alg}\mathscr{L}$ the set $\{T \in L(H): P^{\perp}TP = 0$ for all P in $\mathscr{L}\}$. $\operatorname{alg}\mathscr{L}$ is said to be a CSL algebra if $\mathscr{L}$ is commutative. By a bimodule M of a subalgebra A of $L(H)$ in $L(H)$, we mean a linear subspace M of $L(H)$ such that $AM \subseteq M$ and $MA \subseteq M$.

Let A be a Banach algebra and M be a (two-sided) Banach A-module. Then the space $C^n(A, M)$ of n-cochains $(n = 1, 2, \dots)$ consists of all bounded n-linear maps from the product of n-copies of A to M. The coboundary operator ∂ from $C^n(A, M)$ to $C^{n+1}(A, M)$ is defined by

$$\begin{aligned}(\partial\varphi)&(a_0, a_1, \dots, a_n)\\ &= a_0\varphi(a_1, \dots, a_n) + \sum_{j=1}^{n}(-1)^j\varphi(a_0, \dots, a_{j-2}, a_{j-1}a_j, \dots, a_n)\\ &\quad + (-1)^{n+1}\varphi(a_0, \dots, a_{n-1})a_n\end{aligned}$$

where $\varphi \in C^n(A, M)$. Let $C^0(A, M) = M$ and $\partial: C^0(A, M) \to C^1(A, M)$ be defined by $\partial h(a) = ah - ha$ $(a \in A,\ h \in M)$.

Let

$$Z^n(A, M) = \{\sigma \in C^n(A, M): \partial\sigma = 0\},$$

$$B^n(A, M) = \{\rho \in C^n(A, M): \rho = \partial\varphi \text{ for some } \varphi \in C^{n-1}(A, M)\};$$

then $Z^n(A, M) \supseteq B^n(A, M)$ and the quotient space $Z^n(A, M)/B^n(A, M)$ is called the n-dimensional cohomology group of A with coefficients in M (for details see [3]).

If A is a subalgebra of $L(H)$ and M is a bimodule of A in $L(H)$, then the commutant of A modulo M is denoted by $C(A, M)$, i.e., $C(A, M) =$

Received by the editors July 3, 1990 and, in revised form, March 23, 1991.
1991 *Mathematics Subject Classification*. Primary 47D25, 47D15, 47B47.
Key words and phrases. Bimodules, commutant, cohomology group.

$\{T \in L(H): TS - ST \in M$ for any S in $A\}$. In [1] Christensen proved that if A is a CSL algebra and M is a σ-weakly closed subalgebra of $L(H)$ containing A, then $H^1(A, A) = H^1(A, M)$; therefore, if $H^1(A, A) = 0$ then $C(A, M) = A + M = M$. Unfortunately there exists a CSL algebra A such that $H^1(A, A) \neq 0$. Thus we may ask the following question: does $C(A, M) = M$ for all CSL algebras A and all σ-weakly closed bimodules M containing A? In [2] it is pointed out that this is true for completely distributive CSL algebras. In this note we show that this is true for all CSL algebras. We also prove a theorem on the cohomology groups of CSL subalgebras of Von Neumann algebra.

The proof of the following lemma is similar to the proof of Theorem 4.1 in [4].

Lemma 1. *Let B be a Von Neumann algebra and $\mathscr{L} \subseteq B$ be a complete commutative projection lattice. Let $A = \operatorname{alg}\mathscr{L} \cap B$ and C be the Von Neumann algebra generated by $\mathscr{L}$. Suppose that M is a σ-weakly closed bimodule of A in $L(H)$. If $\varphi \in C^n(A, M)$ is such that $\partial\varphi$ is equal to zero whenever any of its arguments lies in C then there is an element $\sigma \in C^{n-1}(A, M)$ such that $\varphi - \partial\sigma$ is equal to zero whenever any of its arguments lies in C.*

Lemma 2. *Let A and C be as in Lemma 1 and $\varphi \in C^n(A, B)$. Suppose that φ and $\partial\varphi$ vanish whenever one of their arguments lies in C. Then $\varphi \in C^n(A, A)$.*

Proof. By the assumption and since $\mathscr{L} \subseteq C$, it is easy to verify that $\varphi \in C^n(A, \operatorname{alg}\mathscr{L})$ (see the proof of Lemma 2.2 in [5]), thus $\varphi \in C^n(A, A)$.

Lemma 3. *Let A and B be as in Lemma 1. Then $C(A, A) \cap B = A$.*

Proof. If $T \in C(A, A) \cap B$, then $TE - ET \in A \subseteq \operatorname{alg}\mathscr{L}$ for any E in $\mathscr{L}$; thus $E^\perp(TE - ET)E = E^\perp TE = 0$, which implies $T \in \operatorname{alg}\mathscr{L}$ and therefore $T \in A$.

Theorem 4. *Let A, B, and C be as in Lemma 1 and let M be a σ-weakly closed bimodule of A in $L(H)$.*

(1) *If $B \supseteq M \supseteq A$ then $C(A, M) \cap B = M$.*
(2) *If $\operatorname{alg}\mathscr{L} \supseteq M \supseteq A$ then $C(A, M) \cap B = A$.*
(3) *If $M \supseteq \operatorname{alg}\mathscr{L}$ then $C(A, M) = M$.*

Proof. (1) Let $T \in C(A, M) \cap B$. By Lemma 1, there exists $h \in M$ such that $T - h \in C' \subseteq \mathscr{L}' \subseteq \operatorname{alg}\mathscr{L}$. Hence $T - h \in \operatorname{alg}\mathscr{L} \cap B = A$ and therefore $T = T - h + h \in M$.

(2) and (3) are similar to (1).

Corollary 5. *If $A = \operatorname{alg}\mathscr{L}$ is a CSL algebra then $C(A, M) = M$ for any σ-weakly closed bimodule M of A containing A.*

Theorem 6. *Let A, B, C, and $\mathscr{L}$ be as in Lemma 1. Suppose that M is a σ-weakly closed bimodule of A in $L(H)$.*

(1) *If $B \supseteq M \supseteq A$ then $H^n(A, M) = H^n(A, A)$.*
(2) *If $M \supseteq \operatorname{alg}\mathscr{L}$ then $H^n(A, M) = H^n(A, \operatorname{alg}\mathscr{L})$.*

Proof. We only prove (1). The proof of (2) is similar to that of (1), so we leave it to the reader.

Since $B^n(A, A) \subseteq B^n(A, M)$ and $Z^n(A, A) \subseteq Z^n(A, M)$, we define π to be the canonical map from $H^n(A, A)$ to $H^n(A, M)$.

Let $\sigma \in Z^n(A, M)$. Then $\partial\sigma = 0$ and, by Lemma 1, there exists $\rho \in C^{n-1}(A, M)$ such that $\sigma - \partial\rho$ vanishes whenever any of its arguments lies in C. Define $\sigma_1 = \sigma - \partial\rho$. Then $\partial\sigma_1$ and σ_1 vanish whenever any of their arguments lies in C. By Lemma 2, we know that $\sigma_1 \in Z^n(A, A)$. Thus π is surjective.

In order to complete the proof, we need to show that π is injective.

Let $\sigma \in Z^n(A, A)$ and $\sigma = \partial\varphi$, where $\varphi \in C^{n-1}(A, M)$. We will show that there is $\rho \in C^{n-1}(A, A)$ such that $\sigma = \partial\rho$, and thus π is injective.

(i) If $n = 1$ then $\varphi \in C^0(A, M) = M$, and for any a in A we have $\sigma(a) = \partial\varphi(a) = a\varphi - \varphi a$. Since E and $\sigma(E)$ are in A, for any E in $\mathscr{L}$ we have $0 = E^\perp\sigma(E)E = E^\perp(E\varphi - \varphi E)E = -E^\perp\varphi E$, which implies $\varphi \in \operatorname{alg}\mathscr{L}$; therefore, $\varphi \in \operatorname{alg}\mathscr{L} \cap M \subseteq \operatorname{alg}\mathscr{L} \cap B = A$.

(ii) Suppose that $n \geq 2$. Since $\partial\sigma = 0$, by Lemma 1, there exists φ_1 in $C^{n-1}(A, A)$ such that $\sigma - \partial\varphi_1$ vanishes whenever any of its arguments lies in C. Again by Lemma 1, for $\varphi - \varphi_1$ there is $\varphi_2 \in C^{n-2}(A, M)$ such that $\varphi - \varphi_1 - \partial\varphi_2$ vanishes whenever any of its arguments lies in C. Using Lemma 2, for $\varphi - \varphi_1 - \partial\varphi_2$ we have that $\varphi - \varphi_1 - \partial\varphi_2 \in C^{n-1}(A, A)$. Let $\rho = \varphi - \partial\varphi_2$. Then $\rho = (\varphi - \varphi_1 - \partial\varphi_2) + \varphi_1 \in C^{n-1}(A, A)$ and $\partial\rho = \partial\varphi = \sigma$.

Corollary 7. *If $A = \operatorname{alg}\mathscr{L}$ is a CSL algebra, then for any σ-weakly closed bimodule M of A which contains A, we have $H^n(A, A) = H^n(A, M)$.*

References

1. E. Christensen, *Derivations of nest algebras*, Math. Ann. **229** (1977), 155–161.
2. Han Deguang, *On A-submodules for reflexive operator algebras*, Proc. Amer. Math. Soc. **104** (1988), 1067–1070.
3. B. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., no. 127, Amer. Math. Soc., Providence, RI, 1972.
4. B. Johnson, R. Kadison, and J. Ringrose, *Cohomology of operator algebras* III. *Reduction to normal cohomology*, Bull. Soc. Math. France **100** (1972), 73–96.
5. E. C. Lance, *Cohomology and perturbations of nest algebras*, Proc. London Math. Soc. (3) **43** (1981), 334–356.

Department of Mathematics, Qufu Normal University, Qufu, 273165, Shandong, People's Republic of China