PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 3, November 1992

# FACTORIZATION OF SINGULAR MATRICES

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We give a necessary and sufficient condition that a singular square matrix $A$ over an arbitrary field can be written as a product of two matrices with prescribed eigenvalues. Except when $A$ is a $2 \times 2$ nonzero nilpotent, the condition is that the number of zeros among the eigenvalues of the factors is not less than the nullity of $A$. We use this result to prove results about products of hermitian and positive semidefinite matrices simplifying and strengthening some known results.

## 1. INTRODUCTION AND STATEMENT OF MAIN RESULT

In [5] the first author showed that if $A$ is a nonscalar invertible $n \times n$ matrix over a field $F$ and if $\beta_1, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2, \dots, \gamma_n$ are elements of $F$, then $A$ can be written as a product $BC$, where the eigenvalues of $B$ and $C$ are $\beta_1, \dots, \beta_n$ and $\gamma_1, \dots, \gamma_n$, respectively, if and only if $\beta_1\gamma_1 \cdots \beta_n\gamma_n = \det A$. In this article we give a necessary and sufficient condition that a singular matrix can be written as a product of two matrices with prescribed eigenvalues.

We now fix some notation and terminology. The set of all $n \times n$ matrices over a field $F$ is denoted by $M_n(F)$. The determinant of a matrix $A$ is denoted by $\det A$. The eigenvalue of a matrix $A$, denoted $\operatorname{Eig} A$, are always repeated according to algebraic multiplicity, i.e., multiplicity as zeros of the characteristic polynomial. The nullity of $A$, i.e., the dimension of the null space, is denoted by $\operatorname{null}(A)$. The transpose of matrix $A$ is denoted by $A^t$. Vectors in $F^{(n)}$ will be understood to be column vectors and are identified with $n \times 1$ matrices.

We now state our main result.

**Theorem 1.** *Let $A$ be an $n \times n$ singular matrix over a field $F$ And let $\beta_j$ and $\gamma_j$ $(1 \leq j \leq n)$ be elements of $F$. If $A$ is not a nonzero $2 \times 2$ nilpotent, then $A$ can be factored as a product $BC$ with $\operatorname{Eig} B = \{\beta_1, \dots, \beta_n\}$ and $\operatorname{Eig} C = \{\gamma_1, \dots, \gamma_n\}$ if and only if the number of zeros $m$ among $\beta_1, \gamma_1, \dots, \beta_n, \gamma_n$ is not less than the nullity of $A$. If $A$ is a nonzero $2 \times 2$ nilpotent then $A$ can be factored as above if and only if $1 \leq m \leq 3$.*

We note that the two extremal cases of this result are known. It was observed by C. R. Johnson (private communication) that an adaptation of the proof in [5] establishes a factorization as above with $m = \operatorname{null} A$. This is included in

Received by the editors December 14, 1990 and, in revised form, April 1, 1991.
1991 *Mathematics Subject Classification.* Primary 15A23, 15A18.

the version of the first author's factorization theorem [5] presented in the book [2]. The case $m = 2n$ is also known [3, 6, 7] (see Theorem 2 below).

## 2. Preliminaries

**Lemma 1.** *If $A$ is a square matrix that is not a scalar multiple of the identity and if $\lambda \in F$, then $A$ is similar to a matrix whose $(1, 1)$ entry is $\lambda$.*

The easy proof is omitted (see also [5]).

**Lemma 2.** *If $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an $n\times n$ matrix where $A$ and $D$ are square matrices and if $A$ is invertible, then* $\operatorname{null}(T) = \operatorname{null}(D - CA^{-1}B)$.

*Proof.* Let $S = D - CA^{-1}B$, then

$$T = \begin{pmatrix} A & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & S \end{pmatrix}.$$

Since the factor on the left is invertible, it follows that

$$\operatorname{rank}(T) = \operatorname{rank}\begin{pmatrix} I & A^{-1}B \\ 0 & S \end{pmatrix} = n - \operatorname{null}(S)$$

and so $\operatorname{null}(T) = \operatorname{null}(S)$. □

**Lemma 3.** *Let $D$ be an invertible $n \times n$ matrix, and $Y$ any $k \times n$ matrix. Then the two $(n+k) \times (n+k)$ matrices*

$$\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & Y \\ 0 & D \end{pmatrix}$$

*are similar.*

*Proof.*

$$\begin{pmatrix} I & YD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & -YD^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & Y \\ 0 & D \end{pmatrix}. \quad \square$$

The next result was first noted in [1]. We give a proof; for another proof see [7].

**Lemma 4.** *If $A$ is a nonzero $2 \times 2$ nilpotent matrix, then $A$ is not a product of two nilpotent matrices.*

*Proof.* By applying a similarity transformation, we may assume, without loss of generality, that $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. If $A = BC$ where $B$ and $C$ are nilpotents, then we must have that $\operatorname{rank} B = \operatorname{rank} C = \operatorname{rank} A = 1$. Furthermore $\mathcal{N}(C) \subseteq \mathcal{N}(A)$, where $\mathcal{N}(T)$ denotes the null space of $T$. Since both $A$ and $C$ have rank one, it follows that $\mathcal{N}(C) = \mathcal{N}(A) = \operatorname{span}\{e_2\}$, where $e_2 = (0, 1)^t$. Similarly $\operatorname{range} B = \operatorname{range} A = \operatorname{span}\{e_2\}$. Therefore $C = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ for some $b$ and $c$ in $F$. But then $BC = 0$, a contradiction. □

We make use of the following result about products of nilpotent matrices. For matrices over the field of complex numbers, it was proved by Wu [7]; for arbitrary fields, two independently obtained proofs are in [3, 6].

**Theorem 2** (The nilpotent factorization theorem). *Let $A$ be a singular square matrix over an arbitrary field. Then $A$ is a product of two nilpotent matrices if and only if $A$ is not a nonzero $2 \times 2$ nilpotent matrix.*

*Proof of Theorem* 1. We employ the following notation. We denote the list $(\beta_1, \dots, \beta_n)$ by $\boldsymbol{\beta}$, the list $(\gamma_1, \dots, \gamma_n)$ by $\boldsymbol{\gamma}$, and $(\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n)$ by $\boldsymbol{\beta} \cup \boldsymbol{\gamma}$. We say that $\boldsymbol{\beta} = \mathbf{0}$, if $\beta_1 = \beta_2 = \cdots = \beta_n = 0$. We also denote the reduced list $(\beta_2, \dots, \beta_n)$ by $\boldsymbol{\beta}'$.

*Necessity.* Assume that $A = BC$ with $\operatorname{Eig} B = \boldsymbol{\beta}$ and $\operatorname{Eig} C = \boldsymbol{\gamma}$. Let $m_1$ and $m_2$ denote the number of zeros among $\{\beta_1, \dots, \beta_n\}$ and $\{\gamma_1, \dots, \gamma_n\}$, respectively. Then

$$\operatorname{null}(A) \le \operatorname{null}(B) + \operatorname{null}(C) \le m_1 + m_2 = m.$$

Furthermore, if $A$ is a $2 \times 2$ nonzero nilpotent, we have $m \ge 1$ by the above and $m \ne 4$ by Lemma 4, so $1 \le m \le 3$.

*Sufficiency.* We proceed by induction on the size of the matrix $A$. The result is trivial for $1 \times 1$ matrices. Next assume that the conclusion of the theorem is true for all square matrices with size less than $n$, and let $A$ be an $n \times n$ singular matrix and $\beta_1, \gamma_1, \dots, \beta_n, \gamma_n$ be elements of $F$, exactly $m$ of which are zero where $m$ satisfies the conditions of Theorem 1. We consider two cases according as $m < n$ or $n \le m \le 2n - 1$; the case $m = 2n$ being already known (Theorem 2).

*Case* 1. $m < n$. In this case, $\boldsymbol{\beta} \ne \mathbf{0}$ and $\boldsymbol{\gamma} \ne \mathbf{0}$, so with no loss of generality we assume that $\beta_1 \ne 0$, $\gamma_1 \ne 0$. The matrix $A$ is nonzero since $\operatorname{null}(A) \le m < n$; thus, by Lemma 1, $A$ is similar to the matrix

$$A_1 = \begin{pmatrix} \beta_1\gamma_1 & y^{\mathrm{t}} \\ x & D \end{pmatrix}$$

where $x, y \in F^{(n-1)}$ and $D \in M_{n-1}(F)$. It suffices to establish the desired factorization for $A_1$. The number of zeros among $\boldsymbol{\beta}' \cup \boldsymbol{\gamma}'$ equals $m$ and, by Lemma 2, $\operatorname{null}(D - \beta_1^{-1}\gamma_1^{-1}xy^{\mathrm{t}}) = \operatorname{null}(A_1) < m$, we may apply the induction hypothesis to establish the existence of matrices $B_0$ and $C_0$ such that $D - \beta_1^{-1}\gamma_1^{-1}xy^{\mathrm{t}} = B_0C_0$ and $\operatorname{Eig} B_0 = \beta'$, $\operatorname{Eig} C_0 = \gamma'$. (We notice that the exceptional $2 \times 2$ case is not present here, since if $n = 3$ and $\beta_2 = \beta_3 = \gamma_2 = \gamma_3 = 0$, then $m = 4$ and so $m > n$.) Now

$$A_1 = \begin{pmatrix} \beta_1 & 0 \\ \gamma_1^{-1}x & B_0 \end{pmatrix} \begin{pmatrix} \gamma_1 & \beta_1^{-1}y^{\mathrm{t}} \\ 0 & C_0 \end{pmatrix}$$

as required.

*Case* 2. $n \le m \le 2n - 1$. We may assume that $\boldsymbol{\beta} \ne \mathbf{0}$, since otherwise we may factor $A^{\mathrm{t}}$ as $A^{\mathrm{t}} = RS$ with $\operatorname{Eig} R = \boldsymbol{\gamma}$, $\operatorname{Eig} S = \boldsymbol{\beta}$, and then obtain $A = S^{\mathrm{t}}R^{\mathrm{t}}$. So we may take $\beta_1 \ne 0$. Furthermore $m \ge n$, so at least one $\gamma_j = 0$ and we therefore take $\gamma_1 = 0$.

The matrix $A$ is similar to the matrix

$$A_1 = \begin{pmatrix} 0 & y^{\mathrm{t}} \\ 0 & D \end{pmatrix}$$

where $y \in F^{(n-1)}$ and $D \in M_{n-1}(F)$. We notice that the number of zeros among $\beta' \cup \gamma'$ is $m - 1 \ge n - 1 \ge \operatorname{null} D$. We consider two subcases.

*Case* 2(a). $D$ is nonsingular. In this case, there exists a rank-one matrix $R$ such that $D-R$ is singular and in case $n=3$, $D-R$ is not nilpotent. (For example, we may take $R$ to agree with $D$ in one row and have all other rows zero, except when $D$ is a $2\times 2$ matrix with zero diagonal, in which case we first apply a similarity transformation to replace $D$ by a diagonal matrix similar to $D$ and then proceed as before.) The number of zeros among $\boldsymbol{\beta}'\cup\gamma'$ is $m-1$ and $m-1\geq n-1\geq \operatorname{null}(D-R)$; therefore, by the induction hypothesis, $D-R=B_0C_0$ with $\operatorname{Eig}B_0=\boldsymbol{\beta}'$ and $\operatorname{Eig}C_0=\gamma'$. The rank-one matrix $R$ can be written in the form $zw^{\mathrm{t}}$ for some $z$ and $w\in F^{(n-1)}$. Let

$$B=\begin{pmatrix}\beta_1 & 0\\ z & B_0\end{pmatrix} \quad\text{and}\quad C=\begin{pmatrix}0 & w^{\mathrm{t}}\\ 0 & C_0\end{pmatrix};$$

then $\operatorname{Eig}B=\boldsymbol{\beta}$, $\operatorname{Eig}C=\gamma$, and

$$BC=\begin{pmatrix}0 & \beta_1 w^{\mathrm{t}}\\ 0 & D\end{pmatrix},$$

which is similar to $A_2$, by Lemma 3.

*Case* 2(b). $D$ is singular. Unless $A$ is a $3\times 3$ nonzero nilpotent and $\beta_2=\gamma_2=\beta_3=\gamma_3=0$, we may apply the induction hypothesis to $D$ to establish the existence of matrices $B_0$ and $C_0$ such that $D=B_0C_0$ and $\operatorname{Eig}B_0=\boldsymbol{\beta}'$ and $\operatorname{Eig}C_0=\gamma'$. The matrix $A_1$ can now be factored as follows:

$$A_1=\begin{pmatrix}\beta_1 & 0\\ 0 & B_0\end{pmatrix}\begin{pmatrix}0 & \beta_1^{-1}y^{\mathrm{t}}\\ 0 & C_0\end{pmatrix}.$$

If $A$ is a $3\times 3$ nilpotent of rank 1, then $A$ is similar to $\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$ and we proceed as before with $D=\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$.

If $A$ is a $3\times 3$ nilpotent of rank 2 and $\beta_2=\beta_3=\gamma_1=\gamma_2=\gamma_3=0$, then $A$ is similar to a $3\times 3$ upper triangular Jordan cell that has the following factorization:

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}0&0&1\\1&\beta_1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\\0&1&0\end{bmatrix}.$$

This ends the proof of Theorem 1. □

## 3. Products of hermitian and positive semidefinite matrices

We use Theorem 1 and the main result in [5] to give short proofs of and slightly generalize results of Radjavi and Wu. The result of part (a) in the next theorem is due to Wu [8]. In [4] Radjavi proved that every complex square matrix with real determinant is a product of four hermitian matrices, so parts (b) and (c) of Theorem 3 below give a generalization.

**Theorem 3.** *Let $A$ be a real or complex square matrix.*

(a) *If $A$ is singular, then $A$ is a product of four positive semidefinite matrices, three of which may be taken to be definite.*

(b) *If* $\det A$ *is a nonzero real number and $A$ is not a scalar multiple of $I$, then $A$ is a product of four hermitian matrices, at least three of which may be taken to be positive definite.*

(c) *If $A=\lambda I$ and* $\det A$ *is real, then $A$ is a product of four hermitian matrices, none of which can be definite unless $\lambda^2$ is real.*

*Remark.* If $A$ is a real matrix, then $A$ is a product of two real hermitian matrices; indeed every matrix over a field is a product of two symmetric matrices over the same field. This is a classical result of Frobenius. Therefore part (b) is of interest mainly for complex matrices.

*Proof of Theorem* 3. Each of the classes of hermitian, positive definite, and positive semidefinite matrices is invariant under congruence $T \mapsto C^*TC$ for invertible $C$. The equation

$$T^{-1}R_1R_2R_3R_4T = (T^{-1}R_1T^{*-1})(T^*R_2T)(T^{-1}R_3T^{*-1})(T^*R_4T)$$

shows that the set of products described in the statement of Theorem 3 is invariant under similarity.

(a) First assume that $A$ is singular. Using standard canonical forms, it is easy to see that $A$ is similar to a direct sum of matrices, each of which has nullity one. Therefore, it suffices to prove the conclusion of the theorem when $\operatorname{null}(A) = 1$. In this case, by Theorem 1, there exist matrices $B$ (respectively, $C$) such that $A = BC$ and the eigenvalues of $B$ are distinct and positive (respectively, nonnegative). Therefore $B$ (respectively, $C$) is similar to a positive definite matrix (respectively, positive semidefinite), so $B = T^{-1}PT$, $C = R^{-1}SR$ with $P$ positive definite, $S$ positive semidefinite, and $T$ and $R$ invertible. The equation

$$A = (T^{-1}PT^{*-1})(T^*T)(R^{-1}R^{*-1})(R^*SR)$$

gives $A$ as a product of three positive definite matrices and one positive semidefinite.

(b) If $\det T$ is a nonzero real and $T$ is not a scalar multiple of $I$, we use the main theorem in [5] to write $A$ as a product of matrices $BC$ where $B$ has $n$ distinct positive eigenvalues and $C$ has $n$ distinct real eigenvalues, at most one of which is negative. As before $B$ is a product of two positive definite matrices and $C$ is a product of a positive definite and a hermitian matrix. Therefore $A$ is a product of four hermitian matrices, at least three of which can be taken positive definite. We note that if $\det A > 0$, the above proof gives $A$ as a product of four positive definite matrices, a result of Ballantine (see [5]).

(c) If $A = \lambda I$, the proof is as given in [4]. Choose a basis $\{e_1, \ldots, e_n\}$ of $F^{(n)}$. Let $U$ be the shift given by $Ue_j = e_{j+1}$ for $1 \le j \le n-1$ and $Ue_n = e_1$. Therefore, $A = U \cdot \lambda U^{\mathrm{t}}$. If $S$ is the hermitian symmetry given by $Se_j = e_{n+1-j}$ and $K$ the operator given by $Ke_j = \bar{\lambda}^{n+1-j}\lambda^j e_{n+1-j}$, then $US$, $S$, $\lambda U^{\mathrm{t}}K^{-1}$, and $K$ are hermitian and $A = (US)S(\lambda U^{\mathrm{t}}K^{-1})K$ is a product of four hermitian matrices.

Furthermore, if $\lambda I$ is a product of four hermitian matrices, at least one of which is positive definite, then $\lambda PH = H_1H_2$ for hermitian $H_1$, $H_2$, $H$, and a positive definite $P$. Now $H_1H_2$ is similar to $H_2H_1 = (H_1H_2)^*$, so $\lambda PH$ is similar to its adjoint $\bar{\lambda}HP$, which in turn is similar to $\bar{\lambda}PH$. Therefore $\lambda^2PH$ is similar to $\lambda\bar{\lambda}PH$. But $PH$ has real eigenvalues, since it is similar to the positive definite matrix $P^{1/2}HP^{1/2}$. Therefore, $\lambda^2$ is real. □

*Remark* 1. If $\lambda^2$ is real, then $\lambda I$ is either hermitian or is a product of three hermitian matrices. This follows from the equation

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

*Remark* 2. The position of the three positive definite matrices in parts (a) and (b) of Theorem 3 is quite arbitrary. This follows from the fact that a product $PH$ of a positive definite $P$ and hermitian (respectively, positive semidefinite) $H$ can be written as a product of two matrices of the same type but in reverse order as $PH = (PHP^*)P^{*-1}$.

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