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A NOTE ON EXTENSIONS OF SEMIGROUPS OF $*$-ENDOMORPHISMS

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ABSTRACT. We present a relatively elementary proof that every E_0-semigroup acting on a type I_∞ factor can be extended to a one-parameter group of $*$-automorphisms acting on a larger type I_∞ factor.

1. INTRODUCTION

An E_0-semigroup is a semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of normal $*$-endomorphisms of the von Neumann algebra $\mathscr{B}(H)$ of all bounded operators on a Hilbert space H, which satisfies $\alpha_t(1) = 1$ for every t and is weak*-continuous in the natural sense (i.e., $\omega(\alpha_t(A))$ should be continuous in t for fixed $A \in \mathscr{B}(H)$ and $\omega \in \mathscr{B}(H)_*$). A basic result in the theory of E_0-semigroups asserts that every E_0-semigroup α can be extended to a one-parameter automorphism group of a larger type I_∞ factor, in that there is a Hilbert space K (separable if H is separable) and a one-parameter group $\tilde{\alpha} = \{\tilde{\alpha}_t : t \in \mathbb{R}\}$ of $*$-automorphisms of $\mathscr{B}(K \otimes H)$ such that

$$1_K \otimes \alpha_t(A) = \tilde{\alpha}_t(1_K \otimes A), \tag{1.1}$$

for every $t \geq 0$, $A \in \mathscr{B}(H)$. This result was asserted by Powers and Robinson [4], but there is a gap in their argument. Subsequently, the result was established in [2, Corollary 5.21] as a consequence of an analysis of the state space of certain C^*-algebras. Unfortunately, that proof is very indirect, and a number of individuals have expressed interest in having a proof that is based on more conventional technology. The purpose of this note is to give a proof of the following result, which is based on a minor variation of the F. and M. Riesz theorem for C^*-dynamical systems [1, Theorem 5.3].

Theorem A. *Let $\alpha = \{\alpha_t : t \geq 0\}$ be a weak*-continuous semigroup of normal $*$-endomorphisms of a von Neumann algebra M having separable predual, which satisfies $\alpha_t(1) = 1$ for every $t \geq 0$. Then there is a nondegenerate normal representation π of M on a separable Hilbert space K and a strongly continuous*

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one-parameter unitary group acting on K *such that*

$$\pi \circ \alpha_t(x) = U_t \pi(x) U_t^*,$$

for every $x \in M$, $t \geq 0$.

We remark that the conclusion remains true if one drops the separability hypothes on M_*, except that of course K need no longer be separable. Theorem A implies the result discussed above in the case where M is a type I_∞ factor $\mathscr{B}(H)$, since in that case π must be a multiple of the identity representation of $\mathscr{B}(H)$, $\pi(A) = 1_K \otimes A$, $A \in \mathscr{B}(H)$.

2. Extending semigroups to groups

Let $\alpha = \{\alpha_t : t \geq 0\}$ be a semigroup of isometric $*$-endomorphisms of a C^*-algebra A. An *extension* of α is a triple (B, β, θ) consisting of a C^*-algebra B, a one-parameter group $\beta = \{\beta_t : t \in \mathbb{R}\}$ acting on B as $*$-automorphisms, and a faithful $*$-homomorphism $\theta : A \to B$, which is equivariant in the sense that

$$\beta_t \circ \theta = \theta \circ \alpha_t, \qquad t \geq 0.$$

The extension (B, β, θ) is said to be *minimal* if $\bigcup_{t\in\mathbb{R}} \beta_t(\pi(A))$ is norm-dense in B.

Remarks. It is obvious that we can modify any extension (B, β, θ) so as to obtain a smaller extension that is minimal. Moreover, there is a natural notion of (equivariant) isomorphism of extensions of a given (A, α), and any two minimal extensions of (A, α) are isomorphic in this natural sense.

Note too that we have made no assertions about continuity of α or β in the time parameter t. Indeed, for the application we require, the C^*-algebraic continuity condition

$$\lim_{t\to 0} \|\alpha_t(a) - a\| = 0$$

fails for most $a \in A$. Nevertheless, if (A, α) *is* continuous in the sense that the above norm limit is 0 for every $a \in A$, then every minimal extension of (A, α) is easily seen to be a C^*-dynamical system.

There is a construction of extensions of semigroups (A, α) that is similar to more familiar constructions for, say, semigroups of isometries acting on Hilbert spaces. Since we do not know a reference for the result we need, we sketch the argument for completeness.

Proposition 2.1. *Every semigroup* (A, α) *of isometric* $*$*-endomorphisms has a minimal extension* (B, β, θ).

Proof. Consider the C^*-algebra C of all bounded A-valued functions $F : \mathbb{R} \to A$ relative to the sup norm

$$\|F\| = \sup_{x\in\mathbb{R}} \|F(x)\|.$$

The set J of all functions $F \in C$ satisfying the condition

$$\lim_{x\to\infty} \|F(x)\| = 0$$

is a closed ideal in C, and we may consider the quotient C^*-algebra $B = C/J$. Let $F \mapsto \dot{F}$ be the natural projection of C onto B.

The real line acts naturally on C by translations and J is translation-invariant; thus we can define a one-parameter group $\beta = \{\beta_t : t \in \mathbb{R}\}$ of automorphisms of B by $\beta_t(\dot{F}) = \dot{F}_t$, where $F_t(x) = F(x+t)$ for $x, t \in \mathbb{R}$. Finally, define a $*$-homomorphism $\theta : A \to B$ by $\theta(a) = \dot{F}^a$ where F^a is the element of B defined by

$$F^a(x) = \begin{cases} \alpha_x(a), & \text{for } x \geq 0, \\ a, & \text{for } x < 0. \end{cases}$$

We leave it for the reader to check that (B, β, θ) is an extension of (A, α), and thus one obtains a minimal extension by passing to an appropriate C^*-subalgebra of B. □

3. Proof of Theorem A

Let $\alpha = \{\alpha_t : t \geq 0\}$ be a weak*-continuous semigroup of unital $*$-endomorphisms of a von Neumann algebra M, and let (B, β, θ) be a minimal extension of (M, α) in the sense of the preceding section. The C^*-algebra B is in this case the norm-closure of a nested family of von Neumann algebras. More precisely, for every $t \in \mathbb{R}$ we can define a unital C^*-algebraic embedding $\theta_t : M \to B$ by $\theta_t = \beta_t \circ \theta$. Thus, we obtain a one-parameter family $\{M_t : t \in \mathbb{R}\}$ of subalgebras of B via $M_t = \beta_t(M)$.

Each M_t is isomorphic to M and can be considered as a von Neumann algebra. The relation $\beta_s \circ \theta = \theta \circ \alpha_s$ for $s \geq 0$ translates into

$$\theta_{t+s} = \theta_t \circ \alpha_s, \qquad s \geq 0, t \in \mathbb{R}.$$

In particular, if $t_1 \geq t_2$ then the formula

$$\theta_{t_1} = \theta_{t_2} \circ \alpha_{t_1 - t_2}$$

implies that $M_{t_1} \subseteq M_{t_2}$. Moreover, since $\alpha_{t_1-t_2}$ is a normal $*$-endomorphism of M, the inclusion $M_{t_1} \subseteq M_{t_2}$ is a *normal* inclusion of von Neumann algebras. Finally, by minimality of (B, β, θ) it follows that the union

$$B_0 = \bigcup_{t > -\infty} M_t$$

is a norm-dense unital $*$-subalgebra of B.

This filtration of B into von Neumann algebras makes it possible to speak of *locally normal* states of B and *locally normal* mappings of B into itself. For example, a bounded linear functional $\rho \in B^*$ is called locally normal if the restriction of ρ to M_t is normal for every $t \in \mathbb{R}$. The locally normal elements of B^* are a norm-closed linear subspace of B^*. Similarly, because every α_t is a normal endomorphism of M it follows that for every $t \in \mathbb{R}$, β_t is a locally normal $*$-automorphism of B in the sense that β_t restricts to a normal $*$-homomorphism of M_s into M_{t+s} for every $s \in \mathbb{R}$.

We will show that there is a locally normal state ω on B such that

3.1.a. *$\omega\restriction_{M_0}$ is a faithful state of M_0, and*

3.1.b. *The representation $\pi : B \to \mathscr{B}(H_\omega)$ obtained from ω via the GNS construction is covariant in the sense that there is a strongly continuous one-parameter unitary group acting on H_ω which satisfies*

$$\pi(\beta_t(x)) = U_t \pi(x) U_t^*,$$

for every $x \in B$, $t \in \mathbb{R}$.

Notice that the representation $\pi_0 : M \to \mathscr{B}(H_\omega)$ given by $\pi_0 = \pi \circ \theta$ is normal because $\omega \circ \theta$ is a normal state of M, it is nondegenerate, and it is faithful because of 3.1.a. The separability of H_ω follows from the fact that if C is a countable set that is strongly dense in the unit ball of M then

$$\bigcup_{n=1}^{\infty} \pi(\beta_{-n} \circ \theta(C))\xi_\omega,$$

ξ_ω denoting the canonical cyclic vector associated with π, is a countable spanning set in H_ω. Thus Theorem A follows from the existence of such a state.

In order to exhibit such a locally normal state, we claim first that every normal state of M_0 can be extended to a locally normal state of B. To see that, choose any normal state ρ_0 of $M_0 = \theta(M)$. By an obvious induction we can find, for every $n \geq 1$, a normal state ρ_n of M_{-n} such that

$$\rho_n \restriction M_{-n+1} = \rho_{n-1}$$

for every $n = 1, 2, \dots$ (note that here we use normality of the inclusion of M_{-n+1} in M_{-n}). Because the family of linear functionals $\{\rho_n\}$ is coherent, we can define a linear functional ρ_∞ on B_0 by

$$\rho_\infty(x) = \lim_{n \to \infty} \rho_n(x),$$

for $x \in M_0 \cup M_{-1} \cup M_{-2} \cup \cdots = B_0$, and it is obvious that ρ_∞ extends by continuity to a (necessarily locally normal) state of B.

In particular, if we start with a *faithful* locally normal state of M, as we may because M has separable predual, then we may conclude that *there is a locally normal state of B that restricts to a faithful state of M_0.*

Next, we observe that if ω is any locally normal state of B and x is any element of B, then the function $f(t) = \omega(\beta_t(x))$ is a continuous function of $t \in \mathbb{R}$. It is clearly enough to check this assertion for elements x in the dense subalgebra B_0, so fix such an x and suppose that x belongs to M_λ. Fix $t_0 \in \mathbb{R}$. To show that f is continuous at t_0 we first choose $T > 0$ large enough that $T + t_0 > 0$. Because $M_\lambda = \beta_\lambda \circ \theta(M)$, we may find an element $a \in M$ such that

$$x = \beta_\lambda \circ \theta(a)$$

and hence for $|h| < T + t_0$ we have

$$\beta_{t_0+h}(x) = \beta_{\lambda-T} \circ \beta_{t_0+T+h} \circ \theta(a) = \beta_{\lambda-T} \circ \theta(\alpha_{t_0+T+h}(a)) = \theta_{\lambda-T}(\alpha_{t_0+T+h}(a)).$$

Hence

$$f(t_0 + h) = \omega(\theta_{\lambda-T}(\alpha_{t_0+T+h}(a))$$

must tend to $f(t_0)$ as $h \to 0$ because $\alpha_{t_0+T+h}(a)$ tends σ-weakly to $\alpha_{t_0+T}(a)$ and $\omega \circ \theta_{\lambda-T}$ is a normal linear functional on M by local normality of ω.

Now by the preceding paragraphs, we may find a locally normal state ω_0 of B that restricts to a faithful state of $\theta(M)$. Let ϕ be any positive function in $L^1(\mathbb{R})$ whose Fourier transform has compact support, and is normalized so that

$$\int_{-\infty}^{+\infty} \phi(t)\, dt = 1.$$

Define a new state ω of B by

$$\omega(x) = \int_{-\infty}^{+\infty} \phi(t)\, \omega_0(\beta_t(x))\, dt, \qquad x \in B.$$

We claim that ω is a locally normal state of B whose restriction to $\theta(M)$ is faithful, and which has *compact spectrum* in the sense that if $f \in L^1(\mathbb{R})$ is any function whose Fourier transform vanishes on the compact support of $\hat{\phi}$ then

$$\int_{-\infty}^{+\infty} f(t)\, \omega(\beta_t(z))\, dt = 0 \tag{3.2}$$

for every $z \in B$. Indeed, ω is locally normal because for every $T > 0$ the linear functional

$$\omega_T(x) = \int_{-T}^{+T} \phi(t)\, \omega_0(\beta_t(x))\, dt$$

is locally normal (by an obvious application of the bounded convergence theorem), and since

$$\|\omega - \omega_T\| \le \int_{-\infty}^{-T} \phi(t)\, dt + \int_{T}^{\infty} \phi(t)\, dt$$

tends to zero as $T \to \infty$, it follows that ω is locally normal.

To see that $\omega \restriction \theta(M)$ is faithful, choose $a \in M$ for which $\omega(a^*a) = 0$, i.e.,

$$\int_{-\infty}^{+\infty} \phi(t)\, \omega_0(\beta_t(\theta(a^*a)))\, dt = 0.$$

Since ϕ is everywhere positive and the integrand is nonnegative and continuous in t, it follows that $\omega_0(\beta_t(\theta(a^*a)))$ vanishes for all real t. Hence it vanishes in particular for $t = 0$, which implies that $a = 0$ because $\omega_0 \circ \theta$ is faithful on M. We may conclude that ω has property 3.1.a.

Finally, property 3.2 follows from the fact that if f is any integrable function on $\mathbb{R}$ whose Fourier transform vanishes on the support of $\hat{\phi}$, then $(f * \phi)\hat{} = \hat{f}\hat{\phi} = 0$, hence $f * \phi = 0$. Thus for every $z \in B$ we have the desired relation

$$\begin{aligned}\int_{-\infty}^{+\infty} f(t)\, \omega(\beta_t(z))\, dt &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t)\, \phi(s)\, \omega(\beta_{t+s}(z))\, dt\, ds \\ &= \int_{-\infty}^{+\infty} f * \phi\,(u)\, \omega(\beta_u(z))\, du = 0.\end{aligned}$$

The preceding discussion implies that the spectrum of the state ω relative to the one-parameter automorphism group β is compact. In particular, there is a $\lambda \in \widehat{\mathbb{R}}$ for which the spectrum of ω is contained in the half-line $[\lambda, +\infty)$. At this point, we are in a setting very close to that of the F. and M. Riesz theorem for C^*-dynamical systems [1, Theorem 5.3], whose conclusion is precisely the relation 3.1.b that we seek. However, there are two ways in which our setting fails to meet exactly the hypotheses of that theorem.

First, Theorem 5.3 of [1] requires that λ be 0 or at least nonnegative, whereas since ω is a state its spectrum is symmetric about the origin in $\widehat{\mathbb{R}}$, and hence the λ that we have is surely negative. Second, $(B, \mathbb{R}, \beta)$ is not a C^*-dynamical system but rather a locally normal automoprhism group acting on a C^*-algebraic inductive limit of von Neumann algebras. The fact that λ is negative poses no difficulty; indeed the proof of [1, Theorem 5.3] carries over intact to the case where the spectrum of ω is contained in any proper half-line of the form $[\lambda, +\infty)$. The fact that $(B, \mathbb{R}, \beta)$ is not a C^*-dynamical system poses no difficulties either, since the essential thing here is that the group $\beta = \{\beta_t\}$ should be *integrable* in the sense that for every $x \in B$ and every $f \in L^1(\mathbb{R})$, there should exist an element $y \in B$ such that

$$\rho(y) = \int_{-\infty}^{+\infty} f(t)\rho(\beta_t(x))\,dt$$

for every locally normal ρ in B^* [3, p. 300]. The existence of such weak integrals

$$y = \int_{-\infty}^{+\infty} f(t)\beta_t(x)\,dt$$

is easily established in the present context using familiar techniques (cf. [1] for example). □

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