A REMARK ON SAKAI'S QUADRATIC RADON-NIKODYM THEOREM

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(Communicated by Theodore W. Gamelin)

Abstract. Sakai's Radon-Nikodym theorem (in a quadratic form) for normal states on a von Neumann algebra is considered. We show that the conclusion of this theorem follows from a much weaker order assumption on involved states.

1. Introduction

Let \( \psi \) be a faithful normal state on a von Neumann algebra \( M \). Sakai's Radon-Nikodym theorem (in a quadratic form) [12] states that if \( \varphi \in M_+^+ \) satisfies \( \varphi \leq l\psi \) for some \( l > 0 \) then there exists a (unique) positive operator \( h \) in \( M \) \( (0 \leq h \leq l^{1/2}) \) such that \( \varphi(x) = \psi(hxh) \), \( x \in M \). We will point out that the same conclusion follows from a much weaker assumption.

In [8, 11] a necessary and sufficient condition for \( \varphi \) to admit a (bounded) quadratic Radon-Nikodym derivative was found. However, in practical applications checking this condition seems difficult. On the other hand, an unbounded quadratic Radon-Nikodym derivative was studied in [13]. So far the following practical and basic question has been untouched: Does the existence of a (bounded) quadratic Radon-Nikodym derivative follow from the assumption on the order determined by the natural cone \( \mathcal{P}_+ \) [1, 2, 6]? Based on the result [5] we will show that the answer is affirmative (even under a much weaker assumption).

2. Main result

Let \( L^p(M) \) be the Haagerup \( L^p \)-space [7], and assume that \( \varphi \), \( \psi \in M_+^+ \) correspond to \( h_\varphi \), \( h_\psi \in L^1(M)_+ \), respectively. The usual assumption \( \varphi \leq l\psi \) in Sakai's theorem of course means \( h_\varphi \leq lh_\psi \) (as \( \tau \)-measurable operator—here, \( \tau \) is the canonical trace on the crossed product \( M \rtimes_{\sigma} \mathbb{R} \)). Let us assume the following weaker condition [3]: for some \( \varepsilon > 0 \) the Connes Radon-Nikodym cocycle \( f(t) = (D\varphi : D\psi)_t \) \( (t \in \mathbb{R}) \) extends to a bounded \( \sup_\varepsilon \|f(z)\| \leq l \), \( \sigma \)-\( \omega \) continuous function on the strip \( -\varepsilon/2 \leq \text{Im} \ z \leq 0 \) that is analytic in the interior.

Received by the editors March 18, 1991 and, in revised form, April 9, 1991.
1991 Mathematics Subject Classification. Primary 46L10, 46L30.
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For a vector \( \xi \in \mathcal{D}(h^\epsilon_p/2) \cap \mathcal{D}(h^\epsilon_q/2) \), we consider the two functions
\[
g(z) = h^\epsilon_p z \xi, \quad h(z) = f(z) h^\epsilon_q z.
\]
Each of them is a bounded continuous function on the strip \(-\epsilon/2 \leq \text{Im } z \leq 0\) that is analytic in the interior. Since \( f(t) = (D\varphi : D\psi)_t = h^\epsilon_p h^\epsilon_q^{-1} \), \( t \in \mathbb{R} \), we have \( g(z) = h(z) \) for \( z = t \in \mathbb{R} \). Uniqueness of analytic continuation shows \( g(-i\epsilon/2) = h(-i\epsilon/2) \), that is,
\[
h^\epsilon_p z \xi = uh^\epsilon_q z \xi, \quad \xi \in \mathcal{D},
\]
with \( u = f(-i\epsilon/2) \in \mathcal{M} \), \( \|u\| \leq l \). Since \( D \) is a common core for the (\( \tau \)-measurable) operators \( h^\epsilon_p/2 \) and \( uh^\epsilon_q/2 \), we conclude that
\[
h^\epsilon_p = uh^\epsilon_q/2, \quad h^\epsilon_q = h^\epsilon_p u^* uh^\epsilon_q/2 \leq \|u\|^2 h^\epsilon_p \leq l^2 h^\epsilon_q.
\]
Furuta's inequality [5] states that, whenever bounded operators \( A, B \) satisfy \( A > B > 0 \), we get
\[
A^{(p+2r)/q} \geq \left(A^r B^p A^r\right)^{1/q}
\]
for \( r \geq 0, \ p \geq 0, \ q \geq 1, \ (1+2r)q \geq p+2r \). Since this inequality remains valid for \( \tau \)-measurable operators (see the next section), with \( p = 1/\epsilon, \ r = p/2, \ q = 2 \) we get
\[
(h^\epsilon_p/2 h^\epsilon_q h^\epsilon_q/2)^{1/2} \leq l^{1/\epsilon} h^\epsilon_q.
\]
Notice that
\[
\begin{cases}
(h^\epsilon_p/2 h^\epsilon_q h^\epsilon_q/2)^{1/2} = |h^\epsilon_p/2 h^\epsilon_q/2|, \\
\text{tr}(h^\epsilon_p/2 h^\epsilon_q/2 x) = \text{tr}(x h^\epsilon_p/2 h^\epsilon_q/2) = (x h^\epsilon_p/2 h^\epsilon_q/2)_{L^2(M)}, \quad x \in \mathcal{M}.
\end{cases}
\]
Lemma 1 [8, Theorem A]. There exists a (unique) positive operator \( h \) in \( M \) such that \( \varphi(x) = \psi(h x h) \), \( x \in \mathcal{M} \), if and only if the absolute value part \( |\chi_\varphi| \) of the polar decomposition of \( \chi_\varphi = (\xi_\varphi, \xi_\psi) \in \mathcal{M}_* \) satisfies \( |\chi_\varphi| \leq l \psi \) for some \( l > 0 \). Furthermore, in this case \( h \) is exactly \( \|D\chi_\varphi : D\psi\|_{-1/2} \) (so that \( 0 \leq h \leq l^2 \)).

In \( L^p \)-space languages the vectors \( \xi_\varphi, \xi_\psi \in \mathcal{P}_\varphi \) are \( h^\epsilon_p/2, \ h^\epsilon_q/2 \in L^2(M)_+ \), respectively. Hence (3) shows that \( \chi_\varphi \in \mathcal{M}_* \), \( |\chi_\varphi| \in \mathcal{M}_*^+ \) correspond to \( h^\epsilon_p/2 h^\epsilon_q/2 \in L^1(M) \), \( (h^\epsilon_p/2 h^\epsilon_q/2)^{1/2} \in L^1(M)_+ \), respectively. Therefore, (2) means \( |\chi_\varphi| \leq l^{1/\epsilon} \psi \), and Lemma 1 shows the main result of the article.

Theorem 2. Let \( \psi \) be a faithful normal state on a von Neumann algebra \( M \). Assume that \( \varphi \in M^+_\sigma \) satisfies: for some \( \epsilon > 0 \), \( f(t) = (D\varphi : D\psi)_t \), \( t \in \mathbb{R} \) extends to a bounded \( (\sup_z \|f(z)\| \leq l) \), \( \sigma \)-\( \omega \) continuous function on the strip \( -\epsilon/2 \leq \text{Im } z \leq 0 \) that is analytic in the interior. Then there exists a unique positive operator \( h \) in \( M \) \( (0 \leq h \leq l^{1/\epsilon} 1) \) such that \( \varphi(x) = \psi(h x h) \), \( x \in \mathcal{M} \).

For \( \epsilon = 1 \) the theorem is exactly the usual version of Sakai's theorem. When \( \epsilon = 1/2 \), the assumption is equivalent to \( L^2 \xi_\varphi - \xi_\varphi \in \mathcal{P}_\varphi \) (as was shown in [2]). We thus have shown
Corollary 3. Assume that the unique implementing vectors $\xi_\varphi$, $\xi_\psi$ in the natural cone $\mathcal{P}^n$ satisfy $l\xi_\psi - \xi_\varphi \in \mathcal{P}^n$. Then there exists a unique positive operator $h$ in $M$ ($0 \leq h \leq 1$) such that $\varphi(x) = \psi(hxh)$, $x \in M$.

The author does not know (and doubts) if the assumption $l\xi_\psi - \xi_\varphi \in \mathcal{P}^n$ guarantees the existence of a bounded Radon-Nikodym derivative in a Jordan form, i.e., $k \in M_+$ satisfying $\varphi(x) = \psi(kx + \lambda k)$, $x \in M$ (see [8, Proposition 3.2.6; 9, Theorem 1.9]). If the answer is affirmative, then we would obtain a different proof of Corollary 3 (because of [8, Proposition 3.2.7]). On the other hand, starting from the same assumption, Araki [1, Corollary, p. 334] showed the following “vector version”: there exists a positive operator $k \in M$ satisfying $\xi_\varphi = k\xi_\psi + \int k\xi_\psi$. Based on Araki’s result (and without using Furuta’s inequality) one can prove Corollary 3 (by making use of techniques in [8, 10, 11]). However, the proof presented in the article (valid under the much weaker assumption $h^e_\varphi \leq l^2h^e_\psi$) seems easier and more natural.

3. Furuta’s inequality for $\tau$-measurable operators

Here we show that (1) remains valid for $\tau$-measurable operators $A \geq B \geq 0$.

Using the spectral projections $\{e_n\}$ of $A$, we set

$$A_n = e_nA(e_nA) \geq B_n = e_nBe_n \quad (n = 1, 2, \ldots).$$

Since $A_n \geq B_n$ are bounded, (1) implies

$$A^{(p+2r)/q} \geq A_n^{(p+2r)/q} \geq (A_n^A A_n^A)^{1/q}.$$  \hfill (4)

Choose and fix $t > 0$. Let $\mu_t(\cdot)$ be the “$t$th” singular number (see [4] for details). We estimate

$$\mu_t(B - B_n) = \mu_t(B(1 - e_n) + (1 - e_n)Be_n)
\leq \mu_{t/2}(B(1 - e_n)) + \mu_{t/2}((1 - e_n)Be_n)
\leq 2\mu_{t/2}(B(1 - e_n))
\leq 2\mu_{t/4}(B^{1/2})\mu_{t/4}(B^{1/2}(1 - e_n))$$

(note $\mu_{t/4}(B^{1/2}) < +\infty$ since $B$ is $\tau$-measurable)

$$= 2\mu_{t/4}(B^{1/2})\mu_{t/4}(B^{1/2}(1 - e_n))
= 2\mu_{t/4}(B^{1/2})\mu_{t/4}((1 - e_n)B(1 - e_n))^{1/2}
\leq 2\mu_{t/4}(B^{1/2})\mu_{t/4}((1 - e_n)A(1 - e_n))^{1/2} \quad (\text{since } 0 \leq B \leq A)
= 2\mu_{t/4}(B^{1/2})\mu_{t/4}(A - A_n)^{1/2}.$$  \hfill (5)

When $n \to +\infty$, $A_n \to A$ in measure, hence, $\mu_{t/4}(A - A_n) \to 0$. From the above estimate, when $n \to +\infty$, $\mu_t(B - B_n) \to 0$ (for each $t > 0$). We thus know $B_n \to B$ in measure. Thanks to Tikhonov’s result [14] we conclude that

$$(A^A_n B^A_n A^A_n) \to (A^A B^A A^A)^{1/q}$$

in measure. Therefore, by letting $n \to +\infty$ in (4), we get (1) for $\tau$-measurable operators.

References


5. T. Furuta, \( A \geq B \geq 0 \) assures \( (B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q} \) for \( r \geq 0, \ p \geq 0, \ q \geq 1 \) with \( (1 + 2r)q \geq p + 2r \), Proc. Amer. Math. Soc. 101 (1987), 85–88.


