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# ON INVERTIBLE HYPERCYCLIC OPERATORS

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ABSTRACT. Let $A$ be an invertible (bounded linear) operator acting on a complex Banach space $\mathscr{X}$. $A$ is called *hypercyclic* if there is a vector $y$ in $\mathscr{X}$ such that the orbit $\mathrm{Orb}(A; y) := \{y, Ay, A^2y, \dots\}$ is dense in $\mathscr{X}$. ($\mathscr{X}$ is necessarily separable and infinite dimensional.)

**Theorem 1.** *The following are equivalent for an invertible operator $A$ acting on $\mathscr{X}$*: (i) *$A$ or $A^{-1}$ is hypercyclic*; (ii) *$A$ and $A^{-1}$ are hypercyclic*; (iii) *there is a vector $z$ such that* $\mathrm{Orb}(A; z)^- = \mathrm{Orb}(A^{-1}; z)^- = \mathscr{X}$ (*the upper bar denotes norm-closure*); (iv) *there is a vector $y$ in $\mathscr{X}$ such that*

$$[\mathrm{Orb}(A; y) \cup \mathrm{Orb}(A^{-1}; y)]^- = \mathscr{X}.$$

**Theorem 2.** *If $A$ is not hypercyclic, then $A$ and $A^{-1}$ have a common nontrivial invariant closed subset.*

## 1. INTRODUCTION

Only one point is new in Theorem 1. Indeed, the equivalence between (i) and (ii) is already contained in [4] (see also [1] for a different proof). Moreover, if $A$ (and, therefore, $A^{-1}$) is hypercyclic, then its set of hypercyclic vectors are $G_\delta$-dense subsets of $\mathscr{X}$. Since the intersection of two $G_\delta$-dense subsets is a $G_\delta$-dense subset, we deduce that (i) (or (ii)) implies (iii) (same references). On the other hand, the implication (iii) $\Rightarrow$ (i), (ii), and (iv) is trivial. Thus, (i), (ii), and (iii) are equivalent statements, and it only remains to show that (iv) $\Rightarrow$ (i) or (ii).

The class of hypercyclic operators acting on $\mathscr{X}$ is rather large, at least for the case when $\mathscr{X}$ is a Hilbert space. The interested reader is referred to [1–3] for the analysis of this class.

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The first named author passed away in the Fall of 1991, and he is sorely missed by his friends and colleagues. (P.J., ed.)

## 2. Proof of Theorem 2

The second theorem is an unpublished result [4, Theorem 2.15]. Since $A$ is not hypercyclic, neither is $A^{-1}$. Let $y$ be any nonzero vector in $\mathscr{X}$ and let

$$\mathscr{S} = [\mathrm{Orb}(A\,; y) \cup \mathrm{Orb}(A^{-1}\,; y)]^{-}.$$

The set $\mathscr{S}$ is closed, contains a nonzero vector, and it is invariant under $A$ and $A^{-1}$. If $\mathscr{S} \neq \mathscr{X}$, then $\mathscr{S}$ is the required common invariant subset, and the theorem is proved.

Suppose $\mathscr{S} = \mathscr{X}$; then $\mathrm{Orb}(A\,; y)$ is not discrete (otherwise $\mathrm{Orb}(A^{-1}\,; y)^{-} = \mathscr{X}$, contrary to our assumption). So, there is a nonzero vector $w$ and positive integers $n_i \to \infty$ such that $\|w - A^{n_i}y\| \to 0$ $(i \to \infty)$. Let

$$\mathscr{J} = \{x \in \mathscr{X}\colon \|x - A^{r_i}y\| \to 0 \ (i \to \infty) \text{ for some increasing sequence } \{r_i\} \text{ of positive integers}\}.$$

Then $\mathscr{J} \neq \mathscr{X}$ since $\mathscr{J} \subset \mathrm{Orb}(A\,; y)^{-} \neq \mathscr{X}$. Also, $\mathscr{J}$ contains a nonzero $w$ and is invariant under $A$ and $A^{-1}$. (Indeed, if $x \in \mathscr{J}$, then

$$\begin{aligned}\|A^{-1}x - A^{r_i}A^{-1}y\| &= \|A^{-1}x - A^{r_i - 1}y\| \\ &\le \|A^{-1}\| \cdot \|x - A^{r_i}y\| \to 0 \qquad (i \to \infty),\end{aligned}$$

so that $A^{-1}x \in \mathscr{J}$.)

It will be shown that $\mathscr{X} \backslash \mathscr{J}$ is open. Let $v \in \mathscr{X} \backslash \mathscr{J}$; then there is an $\varepsilon > 0$ and a positive integer $N$ such that $\|v - A^n y\| > \varepsilon$ for all $n \geq N$. It follows that $\mathscr{B}(v\,, \varepsilon) := \{x \in \mathscr{X}\colon \|v - x\| < \varepsilon\} \subset \mathscr{X} \backslash \mathscr{J}$.

Hence, $\mathscr{J}$ is a nontrivial closed subset of $\mathscr{X}$ invariant under $A$ and $A^{-1}$.

*Remark.* It follows from the same proof that if $A$ is invertible, but not hypercyclic, and $A^{n_i}y$ converges in the norm to a nonzero vector $w$ for some increasing sequence of integers $n_i$, then $A^{-1}$ has a nontrivial invariant closed subset.

## 3. Proof of Theorem 1

(iv) $\Rightarrow$ (i) or (ii). By hypothesis,

$$\mathscr{X} = [\mathrm{Orb}(A\,; y) \cup \mathrm{Orb}(A^{-1}\,; y)]^{-}.$$

The Baire Category Theorem indicates that either $\mathrm{Orb}(A\,; y)^{-}$ or $\mathrm{Orb}(A^{-1}\,; y)^{-}$ has nonempty interior. If $\mathrm{Orb}(A\,; y)^{-}$ is nowhere dense, then $\mathrm{Orb}(A^{-1}\,; y)^{-} = \mathscr{X}$, and, therefore, $A^{-1}$ is hypercyclic.

Assume interior $\mathrm{Orb}(A\,; y)^{-} \neq \varnothing$ and define $\mathscr{J}$ exactly as above; then $\mathscr{J}$ is a closed subset of $\mathscr{X}$ invariant under $A$ and $A^{-1}$. Clearly, there is a positive integer $p$ and $\delta > 0$ such that $\mathscr{B}(A^p y\,; \delta) \subset \mathrm{Orb}(A\,; y)^{-}$. It follows that

$$\|A^p y - A^{r_i}y\| \to 0 \qquad (i \to \infty)$$

for some increasing sequence $\{r_i\}$ with $r_i \geq p \geq 0$ for all $i \geq 0$, so that $A^p y \in \mathscr{J}$. Since $\mathscr{J}$ is invariant under $A^{-1}$, $y = A^{-p}(A^p y) \in \mathscr{J}$, and, therefore,

$$\mathrm{Orb}(A\,; y)^{-} \supset \mathscr{J} = [\mathrm{Orb}(A\,; y) \cup \mathrm{Orb}(A^{-1}\,; y)]^{-} = \mathscr{X},$$

whence we conclude that $A$ is hypercyclic.

The proof of Theorem 1 is now complete.

## NOTE ADDED IN PROOF

The present proofreading was done by the editor who notes with sorrow that the first named author D. A. H. died of cancer while this paper was in press.

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