WEIGHTED NORM INEQUALITIES
FOR BOCHNER-RIESZ OPERATORS
AND SINGULAR INTEGRAL OPERATORS

XIANLIANG SHI AND QIYU SUN

(Communicated by J. Marshall Ash)

ABSTRACT. Weighted norm inequalities for the Bochner-Riesz operator at the critical index \( \frac{1}{2}(n-1) \) are investigated. We also give some weighted norm inequalities for a class of singular integral operators introduced by Fefferman and Namazi.

1. Introduction and statements

The Bochner-Riesz operators in \( \mathbb{R}^n \) are defined as
\[
(T_{\lambda}^R f)^\sim(x) = \left(1 - R^2|x|^2\right)^{\lambda/2} \hat{f}(x)
\]
and the associated maximal operator is defined as
\[
T_{\lambda}^* f(x) = \sup_{R > 0} |T_{\lambda}^R f(x)|
\]
for \( \lambda > 0 \), where \( \sim \) denotes the Fourier transform. It is well known by the works of Carleson and Sjölin [4], Fefferman [8, 9], Tomas [19], and Christ [6] that \( T_{\lambda}^* \) is bounded on \( L^p(\mathbb{R}^n) \) if and only if \( \frac{1}{p} - 1/2 \leq (1 + 2\lambda)/2n \) provided \( \lambda > 0 \) in dimension 2 and \( \lambda \geq (n-1)/2(n+1) \) in dimension greater than two. Rubio [16] and Hirschman [12] studied the weighted norm inequality for the Bochner-Riesz operator \( T_{\lambda}^* \) and showed that \( T_{\lambda}^* \) is bounded on \( L^2(|x|^a) \) provided \( |a| < 1 + 2\lambda < n \). In 1988 Andersen [1] gave a sufficient condition and a necessary condition on radial weight \( w(|x|) \) such that the inequality
\[
\int_{\mathbb{R}^n} |T_{\lambda}^R f(x)|^2 w(|x|) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(|x|) \, dx
\]
holds for all radial functions \( f \) in \( L^p(w(|x|)) \).

Notice that the Bochner-Riesz operator is a summation operator and \( T_{\lambda}^R f(x) \) tends to \( f(x) \) as \( R \) tends to infinity for all Schwartz functions \( f \). Hence it is meaningful to consider the almost everywhere convergence of \( T_{\lambda}^R f \) as \( R \) tends to infinity for some appropriate function \( f \). In 1986 Lu [14] proved that
\[
T_{\lambda}^R f(x) \to f(x) \quad \text{a.e. as } j \to \infty
\]
for all $f \in L^2(|x|^a)$ provided $0 < a < \min(2, 2\lambda) < n - 1$ and $\{R_j\}_{j=1}^{\infty}$ being a Hadamard lacunary sequence, i.e., $\lim_{j \to \infty} (R_{j+1}/R_j) > 1$. In addition we can reduce almost everywhere convergence of the operator $T^R$ to some maximal inequality. In [14] Lu proved

**Theorem L [14].** Let $0 < \lambda < \frac{1}{2}(n - 1)$, $0 < a < \min(2, 2\lambda)$, and $\{R_j\}_{j=1}^{\infty}$ be a Hadamard lacunary sequence. Then

$$\int \left( \sup_j |T^R_k f(x)| \right)^2 |x|^a \, dx \leq C \int |f(x)|^2 |x|^a \, dx.$$  

In 1988 Carbery, Rubio, and Vega proved

**Theorem CRV [3].** Let $|a| < 1 + 2\lambda < n$. Then

$$\int |T^*_k f(x)|^2 |x|^a \, dx \leq C \int |f(x)|^2 |x|^a \, dx.$$  

Hence they improved Theorem L.

On other hand, we observe that if $\lambda$ exceeds the critical index $\frac{1}{2}(n - 1)$ then $T^*_k$ is dominated by a multiple of the Hardy-Littlewood maximal function $M f$ defined by

$$M f(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y)| \, dy,$$

where supremum is taken over all cubes with center $x$ and sides parallel to the coordinate axes. Hence a result of Muckenhoupt [11] showed that $T^*_k$ is bounded on $L^p(w)$ provided $w \in A_p$, i.e.,

$$\left( |Q|^{-1} \int_Q w(x) \, dx \right) \left( |Q|^{-1} \int_Q w(x)^{- (p - 1)^{-1}} \, dx \right)^{p - 1} \leq C$$

holds for all cubes $Q \subset R^n$ with sides parallel to the coordinate axes and some $C$ independent of $Q$. Then a natural question is whether $T^*_{(n-1)/2}$ is bounded on $L^p(w)$ provided $w \in A_p$ and $1 < p < \infty$. In this paper we prove

**Theorem 1.** Let $1 < p < \infty$ and $w \in A_p$. Then

$$\int |T^*_{(n-1)/2} f(x)|^p w(x) \, dx \leq C \int |f(x)|^p w(x) \, dx.$$  

We also observe that $T^*_{(n-1)/2} f$ can be written as

$$\int h(|y|)|y|^{-n} f(x - y) \, dy,$$

where

$$h(t) = (2\pi)^{n/2} 2^{(n-1)/2} \Gamma(\frac{1}{2}(n - 1)) J_{n-1/2}(t)^{1/2},$$

$\Gamma(t)$ denotes the Gamma function and $J_{n/2}(t)$ denotes the Bessel function, which is defined by

$$J_{n/2}(t) = \frac{(t/2)^n}{\Gamma(n + 1/2)\Gamma(n)} \int_0^{\pi/2} \cos(t \sin u)(\cos u)^{2n} \, du.$$  

Therefore the weighted norm inequality for $T^*_{(n-1)/2}$ is closely related to the one for the operator introduced by Fefferman [9] and Namazi [15].
Now let us write the operator introduced by Fefferman and Namazi precisely. Let \( h \in L^\infty([0, \infty)) \) and \( \Omega \) be an integrable function on the unit sphere \( S^{n-1} \) having mean zero, i.e., \( \int_{S^{n-1}} \Omega(x) \, d\sigma(x) = 0 \), \( d\sigma \) is the standard measure on \( S^{n-1} \).

Define
\[
T_\varepsilon f(x) = \int_{|y| > \varepsilon} h(|y|) \Omega \left( \frac{y}{|y|} \right) |y|^{-n} f(x - y) \, dy
\]
for \( \varepsilon > 0 \),
\[
T_0 f(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x),
\]
and the associated maximal operator
\[
T^* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|.
\]

There are many works about the operators \( T_0 \) and \( T^* \) (see [5, 7, 15, 17, 18], etc.). Namazi [15] proved that \( T_0 \) is a bounded operator on \( L^p(R^n) \) for all \( 1 < p < \infty \) when \( \Omega \in L^q(S^{n-1}) \) for some \( q > 1 \), Chen [5] proved that \( T^* \) is also a bounded operator on \( L^p(R^n) \) when \( \Omega \in L^q(S^{n-1}) \) for some \( q > 1 \), and the second author [18] proved that \( T_0 \) (resp. \( T^* \)) is a bounded operator on \( L^2(R^n) \). As weighted norm inequalities for the operators \( T_0 \) and \( T^* \), Duoandikoetxea and Rubio [7] proved that \( T^* \) and \( T_0 \) are bounded operators on \( L^p(w) \) provided \( 1 < p < \infty \), \( w \in A_p \), and \( \Omega \in L^\infty(S^{n-1}) \).

In this paper we prove the following with the complex interpolation method.

**Theorem 2.** Let \( 1 < q \leq \infty \), \( \Omega \in L^q(S^{n-1}) \), \( q(q - 1)^{-1} < p < \infty \), and \( w \in A_{p(1-1/q)} \). Then
\[
\int |T_0 f(x)|^p w(x) \, dx \leq C \int |f(x)|^p w(x) \, dx
\]
holds for all \( f \) in \( L^p(w) \).

**Theorem 3.** Let \( \Omega \in L^\infty(S^{n-1}) \) and \( w \in A_p \). Then
\[
\int |T^* f(x)|^p w(x) \, dx \leq C \int |f(x)|^p w(x) \, dx
\]
holds for all \( f \) in \( L^p(w) \).

Hence we prove Duoandikoetxea and Rubio’s result in another way. The above results are still interesting even when \( h \equiv 1 \) because in [13] \( \Omega \) satisfies an \( L^r \)–Dini condition for some \( r > 1 \).

2. **Some lemmas**

Define the Bochner-Riesz operator \( T^R_z \) by
\[
(T^R_z f)^-(x) = (1 - R^2 |x|^2)^z_+ f^-(x)
\]
for \( \text{Re} z > 0 \). Then we have
Lemma 1 [3]. Let $\text{Re} \, z > 0$ and $k = 0, 1$. Then

$$
\int \sup_{R > 0} \left( \frac{\partial}{\partial z} \right)^k T_z^R f(x) \right\|^2 dx \leq Ce^{c|\text{Im} \, z|}(|\text{Re} \, z|^{-c} + 1) \int |f(x)|^2 \, dx
$$

holds for a constant $C$ independent of $z$.

To prove Theorem 1, we will also use

Lemma 2. Let $\text{Re} \, z > \frac{1}{2}(n - 1)$. Then the inequality

$$
\sup_{R > 0} |T_z^R f(x)| \leq C(\text{Re} \, z - \frac{1}{2}(n - 1))^{-c} e^{c|\text{Im} \, z|} |f(x)|
$$

holds for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and an absolutely positive constant $C$.

Proof. Write

$$
T_z^R f(x) = \frac{2^{n+1/2} \Gamma(z+1)}{\Gamma(n/2)} \int J_{n/2+z}(R^{-1}|y|) \cdot |y|^{-z-n/2} f(x-y) \, dy.
$$

Also we observe from the asymptotic properties of the Bessel function $J_n(t)$ that

$$
|J_{n/2+z}(t)| < Ct^{n/2+z}
$$

when $t \leq 1$ and

$$
|J_{n/2+z}(t)| < Ct^{-1/2}
$$

when $t > 1$. Therefore for $\text{Re} \, z > \frac{1}{2}(n - 1)$ we have

$$
|T_z^R f(x)| \leq C R^{-n/2+\text{Re} \, z} \int |f(x-y)|R^{-n/2-\text{Re} \, z} \, dy
$$

and Lemma 2 holds.

Lemma 3 [11]. For $s \in (1, \infty)$ and $w \in A_s$, there exists a positive number $\delta$ such that $w^{1+\delta} \in A_s$.

Lemma 4 (Three-Circles Theorem). Suppose $F$ is a bounded continuous complex-valued function on the closed strip $S = \{x + iy : 0 \leq x \leq 1\}$ that is analytic in the interior of $S$. If $|F(iy)| \leq m_0$ and $|F(1 + iy)| \leq m_1$ for all $y$, then $|F(x + iy)| \leq m_0^{1-x} m_1^x$ for all $x + iy \in S$.

To prove Theorems 2 and 3, we need to introduce some notation and use some lemmas. Let $\Omega \subset L^q(S^{n-1})$ for some $q > 1$.

Define

$$
T_{z,\varepsilon} f = \frac{\pi^{(z-1)/2} \Gamma((n-z)/2)}{\Gamma(z/2)} |y|^{-n-z} h(|y|) \Omega \left( \frac{y}{|y|} \right) \chi_{|y|>\varepsilon} \ast |y|^{-n-z} \ast f, \quad \varepsilon > 0,
$$

for $-\frac{1}{2}(1-1/q) < \text{Re} \, z < 1$, where $\ast$ is the convolution operation. Denote the kernel function of the operator $T_{z,0}$ by $K_z$. Therefore we have
Lemma 5 [18]. For \(-\frac{1}{2}(1 - 1/q) < \Re z < 1\) and \(k = 0, 1\), the inequality
\[
\int \sup_{\varepsilon > 0} \left( \frac{\partial}{\partial z} \right)^k T_{z, \varepsilon} f(x) \mid^2 \, dx 
\leq C \left( |\Re z - 1|^{-c} + \left| \Re z + \frac{1}{2} \left( 1 - \frac{1}{q} \right) \right|^{-c} \right) e^{c|\Im z|} \int |f(x)|^2 \, dx
\]
holds.

Lemma 6. For \(0 < \Re z < 1\) and \(1 < p < \infty\), the inequality
\[
\int \sup_{\varepsilon > 0} |T_{z, \varepsilon} f(x)|^p \, dx \leq C(|\Re z|^{-c} + |\Re z - 1|^{-c}) e^{c|\Im z|} \int |f(x)|^p \, dx
\]
holds.

Lemma 7. For \(0 < \Re z < 1\), the inequality
\[
\left( R^{-n} \int_{R < |x| < 2R} |K_z(x + y) - K_z(x)|^q \, dx \right)^{1/q} \leq C(z) R^{-n} \left( \frac{|y|}{R} \right)^{\Re z}
\]
holds for all \(R > 0\) and \(|y| < \frac{1}{2}R\), where
\[
C(z) \leq C(|\Re z|^{-c} + |\Re z - 1|^{-c}) \exp(C|\Im z|).
\]

Lemma 8 [13]. Let \(K \in L^1_{\text{loc}}(R^n \setminus \{0\})\), \(q > 1\), and \(\delta > 0\). Suppose
\[
\left( R^{-n} \int_{R < |x| < 2R} |K(x + y) - K(x)|^q \, dx \right)^{1/q} \leq CR^{-n} \left( \frac{|y|}{R} \right)^{\delta}
\]
holds for all \(R > 0\) and \(|y| < \frac{1}{2}R\). Suppose \(T_\theta\) be defined as \(T_\theta f = K * f\) and \(\Omega\) be bounded on \(L^2(R^n)\). Then the operator \(\Omega\) is bounded on \(L^p(w)\) for all \(q(\delta + 1)^{-1} < p < \infty\) and \(w \in A_p(1-\delta, q)\).

Lemma 9. Let \(\Omega \in L^q(S^{n-1})\) for some \(q > n\). For \(n/q < \Re z < 1\), the inequalities
\begin{enumerate}
\item \(|K_z(x)| \leq C(z)|x|^{-n}\),
\item \(|K_z(x + y) - K_z(x)| \leq C(z)|x|^{-n-(\Re z - n/q)}|y|^{\Re z - n/q})
\end{enumerate}
hold for all \(x \neq 0\) and \(|y| \leq \frac{1}{2}|x|\), where
\[
C(z) \leq C(|\Re z - n/q|^{-c} + |\Re z - 1|^{-c}) \exp(C|\Im z|).
\]

Lemma 10 [11]. Let \(K \in L^1_{\text{loc}}(R^n \setminus \{0\})\). Suppose
\begin{enumerate}
\item \(|K(x)| \leq C|x|^{-n}\),
\item \(|K(x + y) - K(x)| \leq C|y|^\delta|x|^{-n-\delta}
\end{enumerate}
hold for all \(x \neq 0\), \(|y| < \frac{1}{2}|x|\), and some \(\delta > 0\). Suppose an operator \(\Omega\) defined as \(\Omega f = K * f\) is bounded on \(L^2(R^n)\). Then the operator \(\Omega^*\) defined as
\[
\Omega^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} k(y) f(x - y) \, dy \right|
\]
is bounded on \(L^p(w)\) provided \(1 < p < \infty\) and \(w \in A_p\).
The proofs of Lemmas 7 and 9 are elementary and the proof of Lemma 6 is similar to the one in [5]. We omit the details here.

3. Proofs of theorems

Let \( p \in (1, \infty), s \in (1, \infty), 0 < \lambda \leq \frac{1}{2}(n - 1), \) and \( q > n. \) Denote \( \alpha(p) = \frac{1}{p} - \frac{1}{s}, \) \( p^1 = p(p - 1)^{-1}, \)

\[
\theta_1(p, s, \lambda) = \begin{cases} 
\frac{2\lambda}{n - 1}, & \text{when } p = s = 2, \\
\frac{\alpha(p)}{\alpha(s)} \cdot \frac{p}{s}, & \text{when } 0 < \frac{\alpha(p)}{\alpha(s)} < \frac{2\lambda}{n - 1}, \ p \neq 2, \\
0, & \text{otherwise};
\end{cases}
\]

\[
\theta_2(p, s, q) = \begin{cases} 
\left(1 + \frac{2n}{q - 1}\right)^{-1}, & \text{when } p = s = 2, \\
\frac{\alpha(p)}{\alpha(s)} \cdot \frac{p}{s}, & \text{when } 0 < \frac{\alpha(p)}{\alpha(s)} \leq \left(1 + \frac{2n}{q - 1}\right)^{-1}, \ p \neq 2, \\
\frac{1}{2} \left(1 - \frac{\alpha(p)}{\alpha(s)}\right) \left(1 - \frac{1}{q}\right) \frac{p}{s} \cdot \frac{q}{n^1}, & \text{when } \left(1 + \frac{2n}{q - 1}\right)^{-1} < \frac{\alpha(p)}{\alpha(s)} < 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Then the following theorems are general versions of Theorems 1 and 3.

**Theorem 4.** Let \( 0 < \lambda \leq \frac{1}{2}(n - 1), |\alpha(p)| < \lambda/(n - 1), \) and \( w \in A_s. \) Then

\[
\int |T_{\alpha}^* f(x)|^p w_{\theta_1(p, s, \lambda)}(x) \, dx \leq C \int |f(x)|^p \omega_{\theta_1(p, s, \lambda)}(x) \, dx.
\]

**Theorem 5.** Let \( \Omega \in L^q(S^{n-1}) \) for some \( q > n \) and \( w \in A_s. \) Then

\[
\int |T^* f(x)|^p \omega_{\theta_2(p, s, q)}(x) \, dx \leq C \int |f(x)|^p \omega_{\theta_2(p, s, q)}(x) \, dx.
\]

By Lemma 3 and the fact \( |x|^a \in A_s \) for \( -n < a < n(s - 1), \) we improve Theorem L.

**Proof of Theorem 4.** Let \( f, h \) be two nonnegative smooth functions with compact support and \( R(x) \) be any arbitrary positive measurable function bounded below and above, i.e., \( R(x)^{-1} \) and \( R(x) \) bounded. Suppose \( w \in A_s \) and \( \varepsilon_1 \) and \( \varepsilon_2 \) are sufficiently small positive constants chosen later, without loss of generality we assume \( \theta_1(p, s, \lambda) > 0. \) Let

\[
p^{-1} = \frac{1}{2}(1 - \theta) + s^{-1}\theta, \quad \lambda = (1 + \theta)\varepsilon_1 + \theta\left(\frac{1}{2}(n - 1) + \varepsilon_2\right), \quad 0 < \theta < 1.
\]

Denote \( s(z)^{-1} = \frac{1}{2}(1 - z) + s^{-1}z \) for \( 0 < \text{Re } z < 1 \) and \( f_0(x) = \exp(-|x|^2). \)
Define
\begin{align*}
  f^z_{\delta_1, \delta_2}(x) &= (f(x) + \delta_1 f_0(x))^{p(z)-1}(w(x) + \delta_2)^{-s-1} f(x), \\
  h^z_{\delta_1}(x) &= (h(x) + \delta_3 f_0(x))^{p(1-s(z))-1} h(x), \\
  w_N(x) &= \begin{cases} 
    w(x), & \text{when } w(x) \leq N, \\
    N, & \text{when } w(x) > N,
  \end{cases}
\end{align*}
and
\[ T_{x, \varepsilon_1, \varepsilon_2, \delta_1, \delta_2, N}^R(f)(x) = T_{x, \varepsilon_1, \varepsilon_2, \delta_1, \delta_2, N}^R(f_{\delta_1, \delta_2}^{-1})(x) h^z_{\delta_1}(x) d x \]
where \( N^{-1}, \delta_i \ (i = 1, 2, 3) \) are small positive numbers.

Hence by Lemma 1 we can show easily
\[ g(z) = \int T_{x, \varepsilon_1, \varepsilon_2, \delta_1, \delta_2, N}^R(f_{\delta_1, \delta_2}^{-1})(x) h^z_{\delta_1}(x) d x \]
is analytic in the strip \( 0 < \text{Re} \ z < 1 \) and continuous on the closed strip \( 0 \leq \text{Re} \ z \leq 1 \). In addition \( g(z) \) is bounded function, hence by Lemma 4 we have
\[ |g(\theta)| \leq C \left( \sup_{t \in \mathbb{R}} |g(it)| \right)^{1-\theta} \left( \sup_{t \in \mathbb{R}} |g(1+it)| \right)^{\theta}. \]
On the other hand by Lemma 1 we get
\[ \sup_{t \in \mathbb{R}} |g(it)| \leq C_{\varepsilon_1, \varepsilon_2} \sup_{t \in \mathbb{R}} e^{-t^2} e^{t|t|} \left( \int |f(x)|^p \ dx \right)^{1/2} \left( \int |h(x)|^{p'} \ dx \right)^{1/2} \]
\[ \leq C_{\varepsilon_1, \varepsilon_2} \|f\|_{p}^{1/2} \|h\|_{p'}^{1/2} \]
and by Lemma 2
\[ \sup_{t \in \mathbb{R}} |g(1+it)| \leq C_{\varepsilon_1, \varepsilon_2} \sup_{t \in \mathbb{R}} e^{-t^2} \left( \int |T_{1+it, \varepsilon_1, \varepsilon_2, \delta_1, \delta_2, N}^R(f_{\delta_1, \delta_2}^{-1})(x)|^s \ dx \right)^{1/s} \]
\[ \times \left( \int |h^z_{\delta_1}(x)|^{s(s-1)} \right)^{s-1/3} \]
\[ \leq C_{\varepsilon_1, \varepsilon_2} \|f\|_{p}^{s-1} \|h\|_{p'}^{s-1(s-1)}. \]
Therefore
\[ |g(\theta)| \leq C_{\varepsilon_1, \varepsilon_2} \|f\|_{p} \|h\|_{p'} \]
where \( C_{\varepsilon_1, \varepsilon_2} \) is independent of \( \delta_1, \delta_2, \delta_3 \), and \( N \).

Write \( g(\theta) \) as
\[ g(\theta) = \int T_{x, \varepsilon_1, \varepsilon_2, \delta, N}^R(f(w + \delta_2)^{-\theta s} h(x) \ dx \]
(2)
\[ = \int T_{x, \varepsilon_1, \varepsilon_2, \delta, N}^R(f(w + \delta_2)^{-\theta s} h(x) \ dx. \]
Notice that for every \( f \in L^{p'} \) there exist two nonnegative smooth function sequences \( \{f^1_n\} \) and \( \{f^2_n\} \) with compact supports such that
\[ \|f^i_n\|_{p'} \leq C \|f\|_{p'}, \quad i = 1, 2, \ n \in \mathbb{N}, \]
and 
\[ \|f_n - f\|_{p'} \to 0, \quad \text{as } n \to \infty. \]

Hence by (1) and (2) we get
\[ \int |T_{\lambda}^{R(x)}(f(w + \delta_2)^{-\theta s^{-1}})(x)|^p (w_N(x) + \delta_2)^{p \theta s^{-1}} dx \leq C \int |f(x)|^p dx. \]

Let \( N \) tend to infinity in the inequality above. Then we have
\[ \int |T_{\lambda}^{R(x)}(f(w + \delta_2)^{-\theta s^{-1}})(x)|^p (w(x) + \delta_2)^{p \theta s^{-1}} dx \leq C \int |f(x)|^p dx. \]

Notice that \( C \) is independent of \( \delta_2 \) and \( R(x) \) being a measurable function bounded below and above. Therefore
\[ (3) \quad \int |T_{\lambda}^* f(x)|^p w^{p \theta s^{-1}}(x) dx \leq C \int |f(x)|^p w^{p \theta s^{-1}}(x) dx \]
holds for all \( \theta < \theta_1(p, s, \lambda) \neq 0 \) by choosing appropriate \( \epsilon_1 \) and \( \epsilon_2 \). By Lemma 3 we can replace \( w \) in (3) by \( w^{1+\delta} \). Hence Theorem 4 holds by choosing \( \theta = \theta_1(p, s, \lambda)/(1 + \delta) \neq 0 \).

**Proof of Theorem 2.** By Lemmas 7 and 8, \( T_z \) is bounded on \( L^p(w) \) provided \( w \in A_{p(1-1/q)} \) and \( \Re z > 0 \). In addition \( T_z \) is bounded on \( L^2(\mathbb{R}^n) \) when \( |\Re z| < \frac{1}{2}(1 - 1/q) \). By the complex interpolation theorem (see [10]) and Lemma 3, \( T_0 \) is bounded on \( L^p(w) \) provided \( p > q(q - 1)^{-1} \) and \( w \in A_{p(1-1/q)} \). Therefore Theorem 2 holds.

**Proof of Theorem 5.** Define
\[ T_z^* f(x) = \sup_{\epsilon > 0} \left| \int_{|y| > \epsilon} K_z(y) f(x - y) \, dy \right|. \]

Then by Lemmas 9 and 10 we get
\[ \int |T_z^* f(x)|^p w(x) dx \leq C(z) \int |f(x)|^p w(x) dx \]
provided \( w \in A_p \) and \( n/q < \Re z < 1 \). In addition the pointwise estimate
\[ \sup_{\epsilon > 0} |T_{z, \epsilon} f(x)| \leq C(z) \mu f(x) + T_z^* f(x) \]
holds for all \( x \) in \( \mathbb{R}^n \), \( n/q < \Re z < 1 \), and
\[ C(z) \leq C(|\Re z - 1|^{-c} + |\Re z - n/q|^{-c})^{-c} e^{c|\Im z|}. \]

Hence we have
\[ (4) \quad \int \sup_{\epsilon > 0} |T_{z, \epsilon} f(x)|^p w(x) dx \leq C(z) \int |f(x)|^p w(x) dx \]
for \( n/q < \Re z < 1 \) and \( w \in A_p \). By (4), Lemmas 3, 5, and 6 we can prove the following in such a way as we prove Theorem 4,
\[ \int \sup_{\epsilon > 0} |T_{z, \epsilon} f(x)|^q w^{(\Re z)n/q}(x) dx \leq C(z) \int |f(x)|^q w^{(\Re z)n/q}(x) dx \]
provided \(0 < \Re z < n/q, \ w \in A_s,\) and \(C(z) \leq C|\Re z|^{-C} \exp(2|\Im z|^2).\)
Hence
\[
\int \sup_{\varepsilon > 0} |T_{\varepsilon} f|_p(x) w^{\theta_2(p,s,q)}(x) \, dx \leq C \int |f(x)|^p w^{\theta_2(p,s,q)}(x) \, dx
\]
holds for \(1 < p < \infty, \ q > n,\) and \(w \in A_s\) and Theorem 5 holds.

**Acknowledgments**
The authors would like to thank the referee for some corrections.

**References**


**Department of Mathematics, Hangzhou University, Hangzhou, Zhejiang 310028, People's Republic of China**

**Current address.** X. Shi: Department of Mathematics, Texas A & M University, College Station, Texas 77843-3368

**E-mail address:** shi@wavelet1.math.tamu.edu

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use