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PERIODICITY OF SOMOS SEQUENCES

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ABSTRACT. Various sequences related to one introduced by Michael Somos are shown to be periodic mod m for every m, although the problem remains open for the original sequence. Some observations are made concerning the prime divisors and the rate of growth of certain sequences.

1. INTRODUCTION

For $k \geqq 4$, by Somos(k) we shall mean the sequence defined by

$$a_0 = a_1 = \cdots = a_{k-1} = 1,$$
$$a_n a_{n-k} = \sum_{i=1}^{[k/2]} a_{n-i} a_{n-k+i} \quad \text{for } n \geqq k.$$

Somos first introduced the sequence Somos(6) and in [2] raised the question whether all the terms are integers. Some history of the problem may be found in Gale [1]. It is now known that the terms of Somos(k) are all integers for $k = 4, 5, 6, 7$, but not for $k = 8$.

Notice that the sequence Somos(k) can also be extended backwards and has a center of symmetry at $n = (k-1)/2$.

By a generalized Somos sequence, we shall mean a sequence where the initial values $a_0, a_1, \ldots, a_{k-1}$ are positive integers and

$$a_n a_{n-k} = P(a_{n-1}, a_{n-2}, \ldots, a_{n-k+1}) \quad \text{for } n \geqq k,$$

where P is a polynomial with positive integer coefficients.

Such a sequence will be said to satisfy condition (H) if when $a_0, a_1, \ldots, a_{k-1}$ are considered as variables, we may write

$$a_n = p_n(a_0, a_1, \ldots, a_{k-1}) / q_n(a_0, a_1, \ldots, a_{k-1}),$$

where p_n is a polynomial with positive integer coefficients, but q_n is just a product of powers of the variables.

If a sequence satisfies condition (H) and if the initial values are all equal to 1, then clearly all terms are integers. This condition was introduced by Dean

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Hickerson to prove that the terms of Somos(6) are integers, so I named it for him.

The relation $a_n = p_n/q_n$ required by condition (H) is trivial for $n < 2k$. Hickerson showed that to verify the condition in general, it is sufficient to check it for $n = 2k$ and to check that p_k is prime to $p_{k+1}, \dots, p_{2k}$. The result for all $n > 2k$ then follows by innduction. On the one hand, we have

$$a_n = p_{n-1}(a_1, \dots, a_k)/q_{n-1}(a_1, \dots, a_k),$$

and on the other hand,

$$a_n = p_{n-k-1}(a_{k+1}, \dots, a_{2k})/q_{n-k-1}(a_{k+1}, \dots, a_{2k}).$$

Substituting the values of $a_k, a_{k+1}, \dots, a_{2k}$, we obtain an expression for a_n whose denominator is a product of powers of the variables and of p_k in the first case and of $p_{k+1}, \dots, p_{2k}$ in the second. Since p_k is prime to $p_{k+1}, \dots, p_{2k}$, the reduced denominator must be a product of powers of the variables.

Condition (H) is known to hold for Somos(k) when $k = 4, 5, 6, 7$. It was checked for $k = 6$ by Hickerson, using a computer program. A similar check was made for $k = 7$ by Benjamin Lotto. The cases $k = 4$ and $k = 5$ can be checked by hand.

2. Two theorems on periodicity

Since the computation of Somos(k) for $4 \leqq k \leqq 7$ involves division, it is not clear that the terms a_n, known to be integers, must form a sequence that is periodic mod m for each modulus m, although this seems to be the case. Indeed, it is not even clear that periodicity that has been observed must continue. But the first theorem gives information about this.

Theorem 1. *If a generalized Somos sequence satisfies condition* (H), *if* $a_0, a_1, \dots, a_{k-1}$ *are prime to* m, *and if*

$$a_{r+n} \equiv a_n \pmod{m} \quad \textit{for } n = 0, 1, \dots, k-1,$$

then the sequence $a_n \pmod{m}$ *has period* r.

Remark. Under the given hypotheses, the terms of the sequence may not all be integers, but they are m-adic integers (that is, have denominators prime to m), and hence may be considered mod m.

Proof. We have, for all n,

$$a_n = p_n(a_0, a_1, \dots, a_{k-1})/q_n(a_0, a_1, \dots, a_{k-1})$$

and

$$a_{r+n} = p_n(a_r, a_{r+1}, \dots, a_{r+k-1})/q_n(a_r, a_{r+1}, \dots, a_{r+k-1}).$$

Here the numerators are congruent mod m, the denominators are congruent mod m, and, by condition (H), the denominators are prime to m. Hence $a_{r+n} \equiv a_n \pmod{m}$.

This theorem enables us to recognize periodicity mod m for many sequences, including Somos(k) for $k = 4, 5, 6, 7$. But can we predict periodicity? The second theorem provides some information.

Theorem 2. *For* $k=4$ *or* $k=5$, Somos(k) *is periodic* mod m *for every* m.

Proof. We shall make use of the fact that, in these sequences, a_i and a_j are relatively prime whenever $|i-j| \leqq k$. The proof of this result is postponed to §3.

Assuming this, it follows that for any prime p the sequence a_n must have at least k numbers prime to p between any two multiples of p. Hence there are infinitely many blocks of length k of terms prime to p. Given any l, we can find two of these blocks that are congruent mod p^l. By Theorem 1, this establishes periodicity mod p^l going forward. But we can also read the sequence backwards, so we have pure periodicity. Periodicity mod m follows by factoring m into prime powers.

This method does not work for $k=6$ or $k=7$. I do not know any proof of periodicity for these cases.

3. Relative primeness in Somos(4) and Somos(5)

For $k=4$ and $k=5$, it is known, and easily proved by induction, is that two terms a_i and a_j in Somos(k) are relatively prime if $|i-j|<k$. Indeed, this was the basis of a simple proof, given by George Bergman, that these sequences have integer terms. But we need the stronger result that a_i and a_j are relatively prime whenever $|i-j| \leqq k$.

For Somos(4), we need to show that a_n is prime to a_{n+4}. We shall prove that

$$a_n a_{n+6}^2 + a_{n+2}^2 a_{n+8} \equiv 0 \pmod{a_{n+4}}.$$

This shows that if a_n is prime to a_{n+4} then a_{n+4} is prime to a_{n+8}. Since a_n is prime to a_{n+4} for $n=0, 1, 2, 3$, the general result follows by induction.

To simplify the notation, let $b_i = a_{n+i}$. Since b_i satisfies the same recursion relation as a_i, we see that

$$\begin{aligned} b_1 b_5 \cdot b_3 b_7 &= (b_2 b_4 + b_3^2)(b_4 b_6 + b_5^2) \\ &= b_2 b_6 b_4^2 + (b_2 b_5^2 + b_3^2 b_6) b_4 + b_3^2 b_5^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} b_1 b_3 \cdot b_5 b_7 &= (b_0 b_4 - b_2^2)(b_4 b_8 - b_6^2) \\ &= b_0 b_8 b_4^2 - (b_0 b_6^2 + b_2^2 b_8) b_4 + b_2^2 b_6^2. \end{aligned}$$

Equating these mod b_4^2 and noticing that $b_3 b_5 \equiv b_2 b_6$, we see that everything drops out except the multiples of b_4. It follows that

$$-(b_0 b_6^2 + b_2^2 b_8) \equiv b_2 b_5^2 + b_3^2 b_6 \pmod{b_4}.$$

But

$$\begin{aligned} b_1 b_2 (b_2 b_5^2 + b_3^2 b_6) &\equiv b_2^2 b_5 \cdot b_1 b_5 + b_1 b_3^2 \cdot b_2 b_6 \\ &\equiv b_2^2 b_5 \cdot b_3^2 + b_1 b_3^2 \cdot b_3 b_5 \\ &\equiv b_3^2 b_5 (b_2^2 + b_1 b_3) \equiv 0 \pmod{b_4}. \end{aligned}$$

Thus $b_2 b_5^2 + b_3^2 b_6 \equiv 0 \pmod{b_4}$, and hence

$$b_0 b_6^2 + b_2^2 b_8 \equiv 0 \pmod{b_4},$$

as we wished to prove.

For Somos(5), we shall prove that

$$a_n a_{n+7} a_{n+8} + a_{n+2} a_{n+3} a_{n+10} \equiv 0 \pmod{a_{n+5}}.$$

This shows that if a_n is prime to a_{n+5} then a_{n+5} is prime to a_{n+10}. Since a_n is prime to a_{n+5} for $n = 0, 1, 2, 3, 4$, the general result follows by induction.

Again put $b_i = a_{n+i}$. Then we see that

$$\begin{aligned} b_2 b_7 \cdot b_3 b_8 &= (b_3 b_6 + b_4 b_5)(b_4 b_7 + b_5 b_6) \\ &= b_4 b_6 b_5^2 + (b_3 b_6^2 + b_4^2 b_7) b_5 + b_3 b_4 b_6 b_7, \\ b_1 b_6 \cdot b_4 b_9 &= (b_2 b_5 + b_3 b_4)(b_5 b_8 + b_6 b_7), \\ &= b_2 b_8 b_5^2 + (b_2 b_6 b_7 + b_3 b_4 b_8) b_5 + b_3 b_4 b_6 b_7, \\ b_1 b_4 \cdot b_6 b_9 &= (b_0 b_5 - b_2 b_3)(b_5 b_{10} - b_7 b_8) \\ &= b_0 b_{10} b_5^2 - (b_0 b_7 b_8 + b_2 b_3 b_{10}) b_5 + b_2 b_3 b_7 b_8. \end{aligned}$$

Forming the alternating sum of these three equations mod b_5^2, we see that all the terms go out except the multiples of b_5. It follows that

$$(b_3 b_6^2 + b_4^2 b_7) - (b_2 b_6 b_7 + b_3 b_4 b_8) \equiv b_0 b_7 b_8 + b_2 b_3 b_{10} \pmod{b_5}.$$

But $b_3 b_6 \equiv b_2 b_7$ and $b_4 b_7 \equiv b_3 b_8$, so that the left side vanishes mod b_5. Hence

$$b_0 b_7 b_8 + b_2 b_3 b_{10} \equiv 0 \pmod{b_5},$$

as we wished to prove.

4. Some properties of the periods

This section presents some results about Somos(4) and Somos(5) that were obtained by calculation, but for which I do not have general proofs.

There is no obvious rule for predicting the period mod p, but knowing this, the period mod p^l seems to follow. Indeed, if $p > 2$, then the period mod p^l is just p^{l-1} times the period mod p. The case $p = 2$ is peculiar. The following periods were observed. For Somos(4), the period is 5 mod 2, 10 mod 4, and $5 \cdot 2^{l-2}$ mod 2^l for $l \geqq 3$. For Somos(5), the period is 6 mod 2, 12 mod 4 and mod 8, and $3 \cdot 2^{l-2}$ mod 2^l for $l \geqq 4$.

For $k = 4$ and $k = 5$, if a prime p divides any term of Somos(k), then the multiples of p are equally spaced. (This is not true for $k = 6$ or $k = 7$.) However, there are many primes that do not occur at all.

Call the interval between two consecutive multiples of p the gap, denoted by g. Since one gap is centered at $(k-1)/2$, it follows that if we find the smallest positive n for which a_n is divisible by p, then $g = 2n - (k-1)$. Thus the gap is odd when $k = 4$ and even when $k = 5$. Conversely, if the gap is given then p divides a_n just when $n \equiv (g + k - 1)/2 \pmod{g}$.

It turns out that the gap is never much larger than p. Specifically, for every prime $p < 2000$ that occurs as a factor of terms in Somos(4) or Somos(5), we have $g < 1.1p + 6$. Roughly speaking, some multiple of g is near p, but it is hard to make this more precise.

If p occurs as a factor in Somos(4) or Somos(5), then the period mod p must be a multiple of g. It also appears that the period is always a divisor of $(p-1)g$.

TABLE 1

		Somos(4)	Somos(5)	
p	Gap	Period	Gap	Period
2	5	5	6	6
3	7	14	8	16
5		16	10	40
7	9	54		20
11	17	170	12	120
13		96		40
17		144		288
19		90		48
23	11	242		198
29		42	34	952
31		135		228
37		171	14	252
41	51	102		1040
43		147	24	1008
47	57	2622		1012
53	53	1378		1456
59	13	754	24	1392
61	35	350	22	132
67		495		440
71	63	4410		2730
73	75	1350	36	1296
79		1482		5460
83	33	2706	16	1312
89		3784	46	4048
97	47	376		138

Table 1 shows, for Somos(4) and Somos(5), the gap and period mod p for each prime $p < 100$, the gap column being left blank if the prime p does not occur as a factor.

No term of Somos(4) or Somos(5) is divisibly by 4. But for $p > 2$, it appears that if the factor p occurs then p^2 will also occur. In general, the gap mod p^2 seems to be p times the gap mod p.

However, there is an exceptional case when $k = 5$ and $p = 73$. Here all terms divisible by p actually have the factor p^2, so that the gap mod p^2 is the same as the gap mod p. If p^i is the smallest power of p to occur as a factor, I think that the gap mod p^{i+l} will be p^l times the gap mod p^i.

5. PERIODICITY OF GENERALIZED SOMOS SEQUENCES

There are a number of generalized Somos sequences that admit an alternative recursion formula not involving division. This furnishes not only a proof that the terms of the sequence are integers, but also that the sequences are periodic mod m for every m. I do not know such an alternative recursion formula for any Somos(k).

We shall consider three sequences, depending on k, in this section. The first two sequences were suggested by Dana Scott, the third by David Gale. In each case, we start with the initial values $a_0 = a_1 = \cdots = a_{k-1} = 1$.

The first sequence, for any $k \geqq 3$, is defined by

$$a_n a_{n-k} = a_{n-1}^2 + a_{n-2}^2 + \cdots + a_{n-k+1}^2 \quad \text{for } n \geqq k.$$

Here we find that

$$\frac{a_n + a_{n-k}}{a_{n-1}a_{n-2}\cdots a_{n-k+1}}$$

is an invariant. The equality of this expression for two consecutive values of n is an immediate consequence of the recursion formula. For $n = k$, the invariant has the value k. This leads to the alternative recursion formula

$$a_n = k a_{n-1}a_{n-2}\cdots a_{n-k+1} - a_{n-k} \quad \text{for } n \geqq k.$$

The second sequence, for odd $k \geqq 5$, is defined by

$$a_n a_{n-k} = a_{n-1}a_{n-2} + a_{n-3}a_{n-4} + \cdots + a_{n-k+2}a_{n-k+1} \quad \text{for } n \geqq k.$$

Here we find that

$$\frac{a_n + a_{n-k-1}}{a_{n-2}a_{n-3}\cdots a_{n-k+1}}$$

is an invariant. For $n = k+1$, this has the value $k-1$. This leads to the recursion formula

$$a_n = (k-1)a_{n-2}a_{n-3}\cdots a_{n-k+1} - a_{n-k-1} \quad \text{for } n \geqq k+1.$$

Notice that this involves an earlier term than the original recursion formula.

The third sequence, for $k \geqq 4$, is defined by

$$a_n a_{n-k} = a_{n-1}a_{n-2} + a_{n-2}a_{n-3} + \cdots + a_{n-k+2}a_{n-k+1} \quad \text{for } n \geqq k.$$

Here we find that

$$\frac{a_n + a_{n-2} + a_{n-k+1} + a_{n-k-1}}{a_{n-2}a_{n-3}\ldots a_{n-k+1}}$$

is an invariant. For $n = k+1$, this has the value $2k-2$. This leads to the recursion formula

$$a_n = (2k-2)a_{n-2}a_{n-3}\cdots a_{n-k+1} - a_{n-2} - a_{n-k+1} - a_{n-k-1} \quad \text{for } n \geqq k+1.$$

Although the invariant in the third case is the most complicated, it was the first to be found. I discovered the invariant for this sequence with $k = 4$ when I was looking for a proof that the sequence has integer terms. Dean Hickerson pointed out that the invariant could be extended to arbitrary k. After that, I found the invariants for the other sequences considered.

6. Rate of growth of Somos sequences

This section is unrelated to the main theme of this paper, which is the periodicity of various sequences mod m. However, the growth patterns of Somos sequences are so extraordinary that they deserve mention. The conclusions here are based on numerical calculations, but proofs are lacking.

Consider some Somos(k), or any variant obtained by altering the initial values to arbitrary positive integers. It appears that there exist positive constants

A, B, C, and a real constant D such that, for all positive and negative values of n, we will have

$$AC^{(n-D)^2} < a_n < BC^{(n-D)^2}.$$

If the initial values are all 1's, or more generally any sequence that reads the same backwards as forwards, then the sequence a_n has a center of symmetry at $(k-1)/2$, so that we must have $D = (k-1)/2$. In other cases, the value of D is not easy to predict. For example, if we modify Somos(4) by putting $a_3 = 3$, then the value of D is approximately -0.003491.

Furthermore, $a_n/C^{(n-D)^2}$ has a periodic oscillation. For Somos(4), the period is about 2.6391. Indeed, I found that

$$a_n/(1.1076302425)^{(n-3/2)^2} - 0.92293$$

has the same sign as

$$\cos(2\pi(n-3/2)/2.6391)$$

for all n with $|n| < 6000$ and remains rather close to 0.141 times this cosine.

Although the constants so chosen may not be the correct asymptotic values, it still seems remarkable that such a choice is possible. However, it appears that even with the best choice of the constants, some further correction of a smaller order will be necessary. I do not know what form this correction will take.

Added in proof

Here are some more recent results which I have heard about. Michael Somos has found a proof of periodicity mod m for Somos(4) and Somos(5) which is quite different from mine (see §2). Clifford S. Gardner has shown that the multiples of a given number in Somos(4) are equally spaced (see §4), and has found an exact expression in terms of theta functions for the oscillations in Somos(4) and Somos(5) (see §6).

References

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