PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 3, November 1992

THE LONG-TIME BEHAVIOR OF GEODESICS IN CERTAIN LEFT-INVARIANT METRICS

J. E. D'ATRI

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. We introduce a new notion to describe the behavior of geodesics as the parameter increases to infinity and obtain quantitative results for a class of left-invariant metrics that includes the Bergman metric on bounded homogeneous domains.

1. INTRODUCTION

Let M be a complete simply connected Riemannian manifold. If M has sectional curvature $K \leq 0$, then there exists a well-developed theory constructing a boundary of M in terms of the asymptotic behavior of geodesics [2, 7, 8]. If M is, in addition, Riemannian symmetric, then this theory is even more developed and includes aspects of harmonic analysis on M and the structure of subgroups of the isometry group of M [9, 13, 14]. There are, however, large classes of manifolds that would seem to be closely related to manifolds of nonpositive curvature, although they themselves do not enjoy this property. For example, every homogeneous nonsymmetric bounded domain has some positive curvature in the Bergman metric [5], some even have focal points [6], and some carry no invariant metric with nonpositive curvature [3]. Yet these spaces all share with the noncompact Hermitian symmetric spaces a description in terms of normal j-algebras. Similarly, all homogeneous simply connected Riemannian spaces of nonpositive curvature can be described in terms of NC-algebras [1] but NC-algebras also define many homogeneous metrics with some positive curvature. The same situation occurs in the subclass of homogeneous spaces of (strictly) negative curvature with respect to the algebraic description given by Heintze [10].

In this note, we introduce a way to describe the long-time behavior (i.e., behavior as t approaches ∞) of geodesics that seem well adapted to the examples

Received by the editors November 29, 1990 and, in revised form, December 24, 1990.
1991 *Mathematics Subject Classification*. Primary 53C22; Secondary 53C30, 53C35, 22E30.
Key words and phrases. Riemannian metric, left-invariant metric, geodesic, bounded homogeneous domain.

described above. The result we prove seems to be new even for Riemannian symmetric spaces of nonpositive curvature and homogeneous spaces of negative curvature.

2. Results

Let M be a complete simply connected Riemannian manifold. Let $\mathscr{A}$ be an m-dimensional distribution on M, so for each $p \in M$, $\mathscr{A}(p)$ is an m-dimensional vector subspace of the tangent space T_pM. Let $\mathscr{B}$ be the orthogonal complementary distribution and let π denote projection on $\mathscr{B}$.

2.1. **Definition.** Suppose $\gamma\colon [0, \infty) \to M$ is a geodesic ray parameterized by arclength. We say γ is asymptotic to $\mathscr{A}$ if $\lim_{t\to\infty} \pi(\gamma'(t)) = 0$.

2.2. **Definition.** Suppose every geodesic ray is asymptotic to $\mathscr{A}$ and amongst all distributions with this property, $\mathscr{A}$ has minimal dimension. Then we say $\mathscr{B}$ is a maximal distribution of transient directions and call $\dim \mathscr{A}$ the spread of M $(\mathrm{sp}(M))$.

2.3. *Remark.* In general, it would probably be useful to modify 2.2 by allowing distributions that are only defined outside a compact subset of M. Then the spread of n-dimensional Euclidean space would be one (consider the radial distribution). The same argument, together with Toponogov's theorem, gives the same result for nonpositive curvature. However, we will now specialize to homogeneous spaces and distributions invariant under a given group action.

Suppose there is a Lie group S of isometries that acts simply transitively on M and suppose further that we restrict to distributions $\mathscr{A}$ that are S invariant. If we pick a base point $b \in M$, we may identify S with M by $S \ni g \to g(b) \in M$. S then inherits a left-invariant Riemannian metric that induces an inner product on the Lie algebra $\mathfrak{s}$. S invariant distributions on M are then in one-to-one correspondence with vector subspaces of $\mathfrak{s}$ and the orthogonal distribution corresponds to the orthogonal complement in $\mathfrak{s}$. Given an S-invariant distribution $\mathscr{A}$ to which all geodesic rays from one particular point of M are asymptotic, the same is true for geodesic rays from all points. Hence we will restrict our consideration to geodesic rays in S emanating from the identity e. If $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{b}$ is the orthogonal decomposition corresponding to the distribution $\mathscr{A}$ and its orthogonal complement $\mathscr{B}$, then the geodesic ray γ emanating from e is asymptotic to $\mathscr{A}$ if and only if

$$\lim_{t\to\infty} [(L_{\exp \gamma(t)})_*^{-1} \gamma'(t)]_{\mathfrak{b}} = 0$$

where the subscript $\mathfrak{b}$ indicates projection on $\mathfrak{b}$ and for $g \in S$, L_g is left multiplication on S. We let $\mathrm{sp}_S(M)$ be the minimal dimension amongst all S-invariant distributions to which every geodesic ray is asymptotic.

2.4. **Theorem.** *Let M be a simply connected Riemannian manifold, S a Lie group of isometries acting simply transitively on M. Let $\mathfrak{s}$ be the Lie algebra of S and $\mathfrak{b} = [\mathfrak{s}, \mathfrak{s}]$. Suppose*

2.4.1. *There is an abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ so that $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{b}$.*

2.4.2. *The adjoint action of $\mathfrak{a}$ on $\mathfrak{b}$ has only real eigenvalues, and there is a set Λ of real-valued linear functionals on $\mathfrak{a}$ so that $\mathfrak{b} = \sum_{\lambda\in\Lambda} \mathfrak{b}_\lambda$ where $\mathfrak{b}_\lambda = \{V \in \mathfrak{b}\colon [H, V] = \lambda(H)V$ for all $H \in \mathfrak{a}\}$.*

2.4.3. *The spaces* $\{\mathfrak{a}, \mathfrak{b}_\lambda : \lambda \in \Lambda\}$ *are mutually orthogonal.*

2.4.4. *There exists* $H_0 \in \mathfrak{a}$ *so that* $\lambda(H_0) > 0$ *for all* $\lambda \in \Lambda$.

Then $\mathrm{sp}_S(M) = \dim \mathfrak{a}$ *and* $\mathfrak{b}$ *defines a maximal* S*-invariant distribution of transient directions.*

2.5. *Remark.* The above conditions are satisfied in the following cases:

2.5.1. M is a bounded homogeneous domain in $\mathbb{C}^n$ with any admissible metric (including the Bergman metric). Here $\mathfrak{s}$ is the normal j-algebra attached to M.

2.5.2. M is a Riemannian symmetric space of noncompact type. Here S is an Iwasawa subgroup of $I(M)$.

2.5.3. M is a homogeneous manifold of negative curvature.

In example 2.5.1, many cases with some positive curvature are included.

In examples 2.5.2 and 2.5.3, $\dim \mathfrak{a}$ is just the rank of M. It is not clear what the rank should be in general in example 2.5.1 (see [4]), but again $\dim \mathfrak{a}$ is a good candidate.

In example 2.5.3, $\dim \mathfrak{a} = 1$. In the case of hyperbolic geometry, this just says that all geodesics approaching a particular boundary point are becoming tangent to one direction (perpendicular to the boundary in the usual representation). We can also broaden 2.5.3 to include all homogeneous manifolds that, in the notation of Heintze [10], satisfy $D_0 > 0$. This includes many examples with some positive curvature.

2.6. *Remark.* One might consider whether this result has any uniformity, that is, given $\varepsilon > 0$, is there a $T > 0$ so that $|\pi(\gamma'(t))| < \varepsilon$ for $t > T$ and all geodesic rays γ (remember, M is homogeneous)? If this were true, we would have an interesting geometric interpretation: if we are at a point p and only consider geodesics that arrive at p after a sufficient lapse of time $(t > T)$, then these geodesics all approach in directions close to $\mathscr{A}_p$. This is certainly false for hyperbolic geometry where one can explicitly compute the dependence of $T = T(v, \varepsilon)$ on unit vectors v and $\varepsilon > 0$.

2.7. *Remark.* Note that we are not claiming that at any one point p of M we actually have a preferred space of directions $\mathscr{A}_p$. In fact, since this result applies to the rank one symmetric spaces of negative curvature, which have transitive isotropy action, this is clearly impossible. However, if M has some reasonable notion of boundary, this does say something about the limiting directions of geodesic rays as they approach that boundary. Further, $\mathrm{sp}(M)$ is an interesting geometric invariant.

2.8. **Corollary.** *Suppose* G *is a semisimple Lie group of noncompact type and finite center. Let* K *be a maximal compact subgroup and* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *the corresponding Cartan decomposition. Let* $\mathfrak{a}$ *be maximal abelian in* $\mathfrak{p}$ *and let* $\mathfrak{q}$ *be the orthogonal complement to* $\mathfrak{a}$ *in* $\mathfrak{p}$ *with respect to the Killing form. Let* $G = SK$, $S = AN$ *be the corresponding Iwasawa decomposition [11]. For* $g \in G$, *let* $\kappa(g)$ *be the factor in* K, *so* $g(\kappa(g))^{-1} \in S$. *Then for* $X \in \mathfrak{p}$,

$$\lim_{t \to \infty} (Ad\, \kappa(\exp tX))(X))_{\mathfrak{q}} = 0$$

where the subscript denotes projection on $\mathfrak{q}$.

3. Proofs

For any Lie group S with left invariant metric, the covariant derivative satisfies

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle \tag{3.1}$$

for any left invariant vector fields X, Y, Z on S. Let $V_1, \dots, V_n$ be a basis for $\mathfrak{s}$. Let γ be a geodesic ray, parameterized by arclength, emanating from the identity $e \in S$. For each $t \geq 0$, define $f_i(t)$ by

$$\gamma'(t) = \sum f_i(t) V_i|_{\gamma(t)}. \tag{3.2}$$

Then

$$0 = \nabla_{\gamma'}\gamma' = \left(\sum_i f_i' V_i + \sum_{i,j} f_i f_j \nabla_{V_i} V_j\right)\Bigg|_{\gamma(t)} \tag{3.3}$$

so for any $V \in \mathfrak{s}$, we get [6]

$$0 = \sum_i f_i' \langle V_i, V\rangle + \frac{1}{2}\sum_{i,j} f_i f_j \left(\langle [V, V_i], V_j\rangle + \langle [V, V_j], V_i\rangle\right). \tag{3.4}$$

Now specialize to S as in the theorem. By property (4), we can choose a basis $\{H_1, \dots, H_r\}$ of $\mathfrak{a}$ such that $\lambda(H_i) > 0$ for all $\lambda \in \Lambda$, all i. Choose an orthonormal basis $\{Y_i : i \in I_\lambda\}$ for each $\mathfrak{b}_\lambda$. We assume the index sets I_λ are disjoint and let $I = \bigcup I_\lambda$. Change notation so that

$$\gamma'(t) = \left(\sum h_i(t) H_i + \sum g_i(t) Y_i\right)\Big|_{\gamma(t)}. \tag{3.5}$$

Using the basis $\{H_i : i = 1, \dots, r\} \cup \{Y_i : i \in I\}$ and $V = H_j$ in (3.3) gives

$$0 = \sum_i h_i' \langle H_i, H_j\rangle + \sum_\lambda \lambda(H_j) \sum_{i \in I_\lambda} g_i^2 \tag{3.6}$$

for each $j = 1, \dots, r$. Since γ is parameterized by arclength, we have

$$1 = \sum_{i,j} h_i h_j \langle H_i, H_j\rangle + \sum_{i \in I} g_i^2. \tag{3.7}$$

The matrix $(\langle H_i, H_j\rangle)$ is positive definite so the functions h_i are bounded; hence so are the functions

$$k_j = \sum_i h_i \langle H_i, H_j\rangle. \tag{3.8}$$

By (3.6), the k_j are also nonincreasing, hence have (finite) limits as $t \to \infty$. The h_i are linear combinations of the k_j, hence also have limits as $t \to \infty$. Thus $\sum g_i^2$ has a limit $\ell \geq 0$ as $t \to \infty$. Suppose $\ell > 0$. Let $\lambda = \min\{\lambda(H_j) : \forall \lambda, j\}$. Choosing $T > 0$ such that $\sum g_i^2 \geq \ell/2$ for $t \geq T$, (3.6) implies $k_j(t) \leq k_j(T) + (T - t)\lambda\ell/2$ for $t \geq T$. This is impossible since k_j is bounded. Hence $\ell = 0$. This proves every geodesic ray asymptotic to the S invariant distribution determined by $\mathfrak{a}$.

From (3.1), it follows that

$$\nabla_H Y = 0, \quad \forall H \in \mathfrak{a}, \ Y \in \mathfrak{s}. \tag{3.9}$$

For geodesics with $\gamma(0) = e$, $\gamma'(0) = H \in \mathfrak{a}$, we find $\gamma'(t) \equiv H$ so $\mathfrak{b}$ defines a maximal S-invariant distribution of transient directions.

To prove the first corollary, note $G/K \approx S$ by the map $xK \to x(\kappa(x))^{-1} = \sigma(x)$. Each $g \in G$ determines an isometry L_g of G/K given by $L_g(xK) = gxK$; this induces a simply transitive action of S. The geodesics through the identity coset of G/K are $\gamma(t) = (\exp tX)K$ for $X \in \mathfrak{p}$. Further, $\gamma'(t) = (L_{\exp tX})_*^{-1}\gamma'(0)$. Then

$$\left(L_{\sigma(\exp tX)}\right)_*^{-1}\gamma'(t) = \left(L_{\kappa(\exp tX)}\right)_*\gamma'(0) = (\operatorname{Ad}\kappa(\exp tX))(X)$$

after making the standard identification of $\mathfrak{p}$ with the tangent space to G/K at the identity coset. One also checks that in identifying G/K with S, $\mathfrak{a} \subset \mathfrak{p}$ corresponds to $\mathfrak{a} \subset \mathfrak{s}$ and fulfills the role of $\mathfrak{a}$ in the theorem.

Acknowledgment

The author thanks Adam Koranyi for corrections.

References

1. R. Azencott and E. N. Wilson, *Homogeneous manifolds with negative curvature.* I, Trans. Amer. Math. Soc. **215** (1976), 323–362.
2. W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, Birkhäuser, Boston, MA, 1985.
3. J. E. D'Atri, *Sectional curvatures and quasi-symmetric domains*, J. Differential Geom. **16** (1981), 11–18.
4. J. E. D'Atri and J. Dorfmeister, *Flat totally geodesic submanifolds of quasisymmetric Siegel domains*, Geom. Dedicata **28** (1988), 321–336.
5. J. E. D'Atri and I. dotti Miatello, *A characterization of bounded symmetric domains by curvature*, Trans. Amer. Math. Soc. **276** (1983), 531–540.
6. J. E. D'Atri and Zhao Yan da, *Geodesics and Jacobi fields in bounded homogeneous domains*, Proc. Amer. Math. Soc. **89** (1983), 55–61.
7. P. Eberlein, *Rigidity problems of manifolds of nonpositive curvature*, Proceedings of a Conference on Global Differential Geometry and Global Analysis, Berlin, Lecture Notes in Math., vol. 1156, Springer-Verlag, New York, Heidelberg, and Berlin, 1985.
8. P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973), 45–110.
9. H. Furstenberg, *Boundaries of Riemannian symmetric spaces*, Symmetric Spaces–short courses presented at Washington University, Dekker, New York, 1972.
10. E. Heintze, *On homogeneous manifolds of negative curvature*, Math. Ann. **211** (1974), 23–34.
11. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
12. S. Kobayashi and K. Nomizu, *Foundations of differential geometry.* II, Interscience, New York, 1969.
13. A. Koranyi, *Harmonic functions on symmetric spaces*, Symmetric Spaces–short courses presented at Washington University, Dekker, New York, 1972.
14. G. Warner, *Harmonic analysis on semi-simple Lie groups.* I, Springer-Verlag, New York, Heidelberg, and Berlin, 1972.

Institute for Advanced Study, Princeton, New Jersey 08540

Current address: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

E-mail address: datri@math.rutgers.edu