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ON DERIVATIONS IN PRIME RINGS AND BANACH ALGEBRAS

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ABSTRACT. Let R be a ring with center $Z(R)$. A mapping $F: R \to R$ is said to be centralizing on R if $[F(x), x] \in Z(R)$ holds for all $x \in R$. The main purpose of this paper is to prove the following result, which generalizes a classical result of Posner: Let R be a prime ring of characteristic not 2, 3, and 5. Suppose there exists a nonzero derivation $D: R \to R$, such that the mapping $x \mapsto [[D(x), x], x]$ is centralizing on R. In this case R is commutative. Combining this result with some well-known deep results of Sinclair and Johnson, we generalize Yood's noncommutative extension of the Singer-Wermer theorem.

PRELIMINARIES

This paper is a continuation of our earlier work [17]. Throughout, R represents an associative ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$, and use the identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. Recall that R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + x\,D(y)$ holds for all $x, y \in R$. A derivation D is inner if there exists $a \in R$, such that $D(x) = [a, x]$ holds for all $x \in R$. An additive mapping D from R to R is called a Jordan derivation if $D(x^2) = D(x)x + x\,D(x)$ holds for all $x \in R$. Obviously, every derivation is a Jordan derivation. The converse is in general not true. Herstein [7] has proved that every Jordan derivation on a prime ring of characteristic not two is a derivation. A brief proof of Herstein's result can be found in [4]. Cursack [6] has generalized Herstein's result on 2-torsionfree (i.e., such that $2x = 0$ implies $x = 0$) semiprime rings (see also [2]). A mapping F from R to R is said to be commuting on R if $[F(x), x] = 0$ holds for all $x \in R$, and is said to be centralizing on R if $[F(x), x] \in Z(R)$ holds for all $x \in R$. For results concerning commuting, centralizing, and related mappings in prime and semiprime rings, we refer to [1, 5, 9, 10, 16, 17] where further references can be found.

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THE RESULTS

A classical result in the theory of centralizing mappings is a theorem of Posner [11], which states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. In our recent paper [17] we have proved that in case there exists a nonzero derivation $D: R \to R$, where R is a prime ring of characteristic different from 2 and 3, such that the mapping $x \mapsto [D(x), x]$ is centralizing on R, R is commutative. Neglecting the fact that in our result we have an additional assumption concerning the characteristic of the ring, we can say that Theorem 2 in [17] generalizes Posner's theorem mentioned above. It is our aim in this paper to generalize Theorem 2 in [17] by proving the following result.

Theorem 1. *Let R be a noncommutative prime ring of characteristic different from 2, 3, and 5. Suppose there exists a derivation $D: R \to R$ such that the mapping $x \mapsto [[D(x), x], x]$ is centralizing on R. In this case $D = 0$.*

Proof. We introduce a mapping $B(\cdot, \cdot): R \times R \to R$ by the relation

$$B[x, y] = [D(x), y] + [D(y), x], \qquad x, y \in R.$$

Obviously, $B(\cdot, \cdot)$ is symmetric (i.e., $B(x, y) = B(y, x)$ for all $x, y \in R$) and additive in both arguments. A routine calculation shows that the relation

$$B(xy, z) = B(x, z)y + xB(y, z) + D(x)[y, z] + [x, z]D(y) \tag{1}$$

holds for all $x, y, z \in R$. We also introduce a mapping f from R to R by $f(x) = B(x, x)$. We have

$$f(x) = 2[D(x), x], \qquad x \in R. \tag{2}$$

Obviously, the mapping f satisfies the relation

$$f(x + y) = f(x) + f(y) + 2B(x, y), \qquad x, y \in R. \tag{3}$$

Throughout the proof, we use the mapping $B(\cdot, \cdot)$ and the relations (1), (2), and (3) without specific reference. The assumption of the theorem can now be written in the form

$$[[f(x), x], x] \in Z(R), \qquad x \in R. \tag{4}$$

First we intend to prove that the mapping $x \mapsto [f(x), x]$ is commuting on R. In other words, we are going to prove that

$$[[f(x), x], x] = 0 \tag{5}$$

holds for all $x \in R$. The linearization of (4) gives

$$\begin{aligned}&[[f(y), x], x] + 2[[B(x, y), x], x] + [[f(x), y], x] + [[f(y), y], x]\\&\quad + 2[[B(x, y), y], x] + [[f(x), x], y] + [[f(y), x], y]\\&\quad + 2[[B(x, y), x], y] + [[f(x), y], y] + 2[[B(x, y), y], y] \in Z(R).\end{aligned}$$

Substituting $-x$ for x in the above relation and comparing the new relation with the above relation, we obtain

$$\begin{aligned}&2[[B(x, y), x], x] + [[f(x), y], x] + [[f(y), y], x] + [[f(x), x], y]\\&\quad + [[f(y), x], y] + 2[[B(x, y), y], y] \in Z(R).\end{aligned} \tag{6}$$

Substituting $2x$ instead of x in (6), comparing the relation so obtained with (6), and using the fact that we have assumed that R is of characteristic not 3, we obtain

(7) $[[f(x), x], y]+[[f(x), y], x]+2[[B(x, y), x], x] \in Z(R), \qquad x, y \in R.$

Substituting x^2 instead of y in (7), one obtains easily $[[f(x), x], x]x \in Z(R)$, $x \in R$, which together with (4) gives

$$[[f(x), x], x][x, y] = 0, \qquad x, y \in R. \tag{8}$$

From (8) and Lemma 1 in [11], one can conclude that for any fixed $x \notin Z(R)$, we have $[[f(x), x], x] = 0$ (note that for any fixed $x \notin Z(R)$ a mapping $y \mapsto [x, y]$ is a nonzero inner derivation), which proves the relation (5). From (5) we obtain

(9) $[[f(x), x], y] + [[f(x), y], x] + 2[[B(x, y), x], x] = 0, \qquad x, y \in R,$

in the same fashion that makes it possible to obtain (7) from (4). Let y be xy in (9). Then using (5) and (9), after some calculations we obtain

$$\begin{aligned}3[f(x), x][y, x] + 2f(x)[[y, x], x]\\ + D(x)[[[y, x], x], x] = 0, \qquad x, y \in R.\end{aligned} \tag{10}$$

Similarly, one obtains the relation

$$\begin{aligned}3[y, x][f(x), x] + 2[[y, x], x]f(x)\\ + [[[y, x], x], x]D(x) = 0, \qquad x, y \in R.\end{aligned} \tag{11}$$

Putting $y = 2D(x)$ in (10) and (11), we arrive at

$$3[f(x), x]f(x) + 2f(x)[f(x), x] = 0, \qquad x \in R, \tag{12}$$

and

$$3f(x)[f(x), x] + 2[f(x), x]f(x) = 0, \qquad x \in R. \tag{13}$$

From (12) and (13) it follows immediately that $5[f(x), x]f(x)+5f(x)[f(x), x] = 0$, $x \in R$, which gives $[f(x), x]f(x) + f(x)[f(x), x] = 0$, $x \in R$, because we have assumed that R is of characteristic not 5. Now $[f(x), x]f(x) = -f(x)[f(x), x]$, $x \in R$, together with (12) gives

$$f(x)[f(x), x] = 0, \qquad x \in R, \tag{14}$$

and

$$[f(x), x]f(x) = 0, \qquad x \in R. \tag{15}$$

From (14) we obtain

$$\begin{aligned}f(x)[f(x), y] + 2f(x)[B(x, y), x]\\ + 2B(x, y)[f(x), x] = 0, \qquad x, y \in R,\end{aligned} \tag{16}$$

using the same approach as in the proof of (7). Put yx instead of y in (16). Then

$$\begin{aligned}0 &= f(x)[f(x), yx] + 2f(x)[B(x, y)x + yf(x) + [y, x]D(x), x]\\ &\quad + 2(B(x, y)x + yf(x) + [y, x]D(x))[f(x), x]\\ &= f(x)[f(x), y]x + f(x)y[f(x), x] + 2f(x)[B(x, y), x]x\\ &\quad + 2f(x)[y, x]f(x) + 2f(x)y[f(x), x] + 2f(x)[[y, x], x]D(x)\\ &\quad + f(x)[y, x]f(x) + 2B(x, y)x[f(x), x] + 2yf(x)[f(x), x]\\ &\quad + 2[y, x]D(x)[f(x), x].\end{aligned}$$

According to (16), one can write $-2B(x, y)[f(x), x]x$ in the above calculation instead of $f(x)[f(x), y]x+2f(x)[B(x, y), x]x$. Now by (5) and (14), we have

$$3f(x)y[f(x), x] + 3f(x)[y, x]f(x) + 2f(x)[[y, x], x]D(x) \\ + 2[y, x]D(x)[f(x), x] = 0,$$

which can be written in the form

$$\begin{aligned} &3[f(x), y][f(x), x] + 3f(x)[y, x]f(x) + 2f(x)[[y, x], x]D(x) \\ &\quad + 2[y, x]D(x)[f(x), x] = 0, \qquad x, y \in R. \end{aligned} \tag{17}$$

Substituting yz for y in (17), after some calculations and similar substitutions as in the proof of (17) we obtain

$$\begin{aligned} &3[f(x), y]z[f(x), x] + 3[f(x), y][z, x]f(x) + 2[f(x), y][[z, x], x]D(x) \\ &\quad + 3f(x)[y, x]zf(x) + 2f(x)[[y, x], x]zD(x) \\ &\quad + 4f(x)[y, x][z, x]D(x) + 2[y, x]zD(x)[f(x), x] = 0 \end{aligned}$$

and in particular for $y = f(x)$,

$$[f(x), x]yD(x)[f(x), x] = 0, \qquad x, y \in R,$$

which gives

$$D(x)[f(x), x] = 0, \qquad x \in R, \tag{18}$$

by primeness of R. In the same fashion one can prove the relation

$$[f(x), x]D(x) = 0, \qquad x \in R, \tag{19}$$

starting from (15). From (18) and (19) one obtains

$$D(x)[f(y), y] + D(y)[f(y), x] + 2D(y)[B(x, y), y] = 0, \qquad x, y \in R, \tag{20}$$

and

$$[f(y), y]D(x) + [f(y), x]D(y) + 2[B(x, y), y]D(y) = 0, \qquad x, y \in R. \tag{21}$$

Substituting xy for x in (20), one obtains

$$\begin{aligned} &3[D(y), x][f(y), y] + 3D(y)[x, y]f(y) \\ &\quad + 2D(y)[[x, y], y]D(y) = 0, \qquad x, y \in R. \end{aligned} \tag{22}$$

Similarly, (21) gives

$$\begin{aligned} &3[f(y), y][x, D(y)] + 3f(y)[x, y]D(y) \\ &\quad + 2D(y)[[x, y], y]D(y) = 0, \qquad x, y \in R. \end{aligned} \tag{23}$$

Now the substitution xz for x in (22) leads to

(24)

$$\begin{aligned} &3[D(y), x]z[f(y), y] + 3[D(y), x][z, y]f(y) + 2[D(y), x][[z, y], y]D(y) \\ &\quad + 3D(y)[x, y]zf(y) + 2D(y)[[x, y], y]zD(y) \\ &\quad + 4D(y)[x, y][z, y]D(y) = 0, \qquad x, y \in R. \end{aligned}$$

Similarly one can prove the relation

(25)
$$3[f(y), y]z[x, D(y)] + 3f(y)[z, y][x, D(y)] + 2D(y)[[z, y], y][x, D(y)] + 3f(y)z[x, y]D(y) + 2D(y)z[[x, y], y]D(y) + 4D(y)[z, y][x, y]D(y) = 0, \qquad x, y \in R,$$

putting zx instead of x in (23). In particular, for $x = 2D(y)$, (24) and (25) reduce to

(26) $$3D(x)f(x)yf(x) + 4D(y)f(x)[y, x]D(x) = 0, \qquad x, y \in R,$$

and

(27) $$3f(x)yf(x)D(x) + 4D(x)[y, x]f(x)D(x) = 0, \qquad x, y \in R.$$

Replacing y by $yD(x)$ in (26), we arrive at

(28)
$$3D(x)f(x)yD(x)f(x) + 4D(x)f(x)[y, x]D(x)^2 + 2D(x)f(x)yf(x)D(x) = 0$$

for all $x, y \in R$. On the other hand, right multiplication of (26) by $D(x)$ leads to

(29) $$3D(x)f(x)yf(x)D(x) + 4D(x)f(x)[y, x]D(x)^2 = 0, \qquad x, y \in R.$$

Combining (28) with (29), we arrive at

(30) $$3D(x)f(x)yD(x)f(x) - D(x)f(x)yf(x)D(x) = 0, \qquad x, y \in R.$$

Similarly, one obtains the relation

(31) $$3f(x)D(x)yf(x)D(x) - D(x)f(x)yf(x)D(x) = 0, \qquad x, y \in R,$$

starting from (27). Combining (30) with (31), one obtains

(32) $$D(x)f(x)yD(x)f(x) = f(x)D(x)yf(x)D(x), \qquad x, y \in R.$$

Now it follows from (32) and Theorem 7 in [2] that for any fixed $x \in R$ we have either $D(x)f(x) = f(x)D(x)$ or $D(x)f(x) = -f(x)D(x)$. In both cases, the relation (31) reduces to $f(x)D(x)yf(x)D(x) = 0$, $x, y \in R$, which gives

(33) $$f(x)D(x) = 0, \qquad x \in R,$$

by primeness of R. Of course we have also

(34) $$D(x)f(x) = 0, \qquad x \in R.$$

From (33) one obtains

(35) $$f(x)D(y) + 2B(x, y)D(x) = 0, \qquad x, y \in R.$$

Substitute yz for y in (35). Then after some calculations and suitable substitutions, we obtain $2B(x, y)[z, D(x)] + [f(x), y]D(z) + 2D(y)[z, x]D(x) + 2[y, x]D(z)D(x) = 0$, $x, y, z \in R$, and in particular, for $z = D(x)$,

(36) $$[f(x), y]D^2(x) + 2[y, x]D^2(x)D(x) = 0, \qquad x, y \in R,$$

since (33) holds. Substituting yz for y in (36) gives

(37) $$[f(x), y]zD^2(x) + 2[y, x]zD^2(x)D(x) = 0, \qquad x, y, z \in R.$$

In particular, for $y = D(x)$ the relation (37) reduces to

$$f(x)yD^2(x)D(x) = 0, \qquad x, y \in R, \tag{38}$$

because (33) and (34) hold. Let us assume that $D^2(a)D(a) \neq 0$ for some $a \in R$. In this case it follows from (38) that $f(a) = 0$. Hence (37) reduces to $[y, a]zD^2(a)D(a) = 0$, which implies $a \in Z(R)$, since we have assumed that $D^2(a)D(a) \neq 0$. In other words, we have proved that $D^2(x)D(x) = 0$ for any $x \neq Z(R)$. We intend to prove that

$$D^2(x)D(x) = 0 \tag{39}$$

for all $x \in R$. Therefore, let x be from $Z(R)$ and let $y \neq Z(R)$. We also have $x+y \neq Z(R)$. We know that $D^2(y)D(y) = 0$ and $D^2(x+y)D(x+y) = 0$, whence it follows $D^2(x)D(x) + D^2(x)D(y) + D^2(y)D(x) = 0$. Substituting $-x$ for x and comparing both relations, we obtain (39). The linearization of (39) gives $D^2(x)D(y) + D^2(y)D(x) = 0$, x, $y \in R$. Substituting yz for y, one obtains easily $D^2(y)[z, D(x)] + [D^2(x), y]D(z) + 2D(y)D(z)D(x) = 0$, x, y, $z \in R$, and in particular, for $z = D(x)$,

$$[D^2(x), y]D^2(x) = 0, \qquad x, y \in R. \tag{40}$$

For any fixed $x \in R$ the mapping $y \mapsto [D^2(x), y]$ is an inner derivation. Hence from (40) and Lemma 1 in [11] it follows that for any fixed $x \in R$ we have either $D^2(x) = 0$ or $D^2(x) \in Z(R)$. In any case $D^2(x) \in Z(R)$ for all $x \in R$, which makes it possible to conclude that left multiplication of (39) by y gives $D^2(x)yD(x) = 0$, x, $y \in R$. From this relation it follows that for any $x \in R$, we have either $D^2(x) = 0$ or $D(x) = 0$ by primeness of R. In any case, $D^2(x) = 0$ for all $x \in R$, which yields $D = 0$ by Theorem 2 in [11]. The proof of the theorem is complete.

We feel that Theorem 1 can be proved without the assumption that R is of characteristic different from 5, but unfortunately we are unable to do it.

Theorem 1 leads to the following conjecture: Let $D: R \to R$ be a derivation, where R is a noncommutative prime ring with suitable characteristic restrictions. Suppose that for some integer n we have $f_n(x) = 0$ for all $x \in R$, where $f_1(x) = [D(x), x]$ and $f_{n+1}(x) = [f_n(x), x]$. In this case $D = 0$. We feel that this conjecture would be hard to prove for arbitrary n, and would almost certainly require higher-powered methods than those used in the proof of Theorem 1.

Singer and Wermer [14] have proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Thomas [15] has proved this result without assuming the continuity of derivation. Yood [18] has extended the Singer-Wermer theorem on noncommutative Banach algebras by proving that every continuous linear derivation D of a Banach algebra A, which satisfies $[D(x), y] \in \mathrm{rad}(A)$ for all x, $y \in A$, where $\mathrm{rad}(A)$ denotes the radical of A, maps A into $\mathrm{rad}(A)$. We now generalize Yood's result as follows.

Theorem 2. *Let A be a noncommutative Banach algebra, and let $D: A \to A$ be a continuous linear Jordan derivation. If $[[[D(x), x], x], x] \in \mathrm{rad}(A)$ for all $x \in A$, then D maps A into $\mathrm{rad}(A)$.*

Proof. By Lemma 3.2 in Sinclair's paper [13], every continuous linear Jordan derivation D of a Banach algebra A leaves the primitive ideals of A invariant. Since the radical of A is the intersection of all primitive ideals, we have $D(\mathrm{rad}(A)) \subset \mathrm{rad}(A)$, which means that there is no loss of generality in assuming that A is semisimple. Since D leaves all primitive ideals invariant, one can introduce for any primitive ideal $P \subset A$ a Jordan derivation $D_P\colon A/P \to A/P$, where A/P is the factor algebra, by $D_P(\hat{x}) = D(x)$, $\hat{x} = x + P$. The factor algebra A/P is prime, since P is a primitive ideal. Hence by Herstein's result D_P is a derivation. The assumption of the theorem $[[[D(x), x], x], x] = 0$, $x \in R$, gives $[[[D_P(\hat{x}), \hat{x}], \hat{x}], \hat{x}] = 0$, $\hat{x} \in A/P$. Hence, in case A/P is noncommutative, we have $D_P = 0$, since all the assumptions of Theorem 1 are fulfilled. It remains to prove that $D_P = 0$ also in the case when A/P is commutative. Johnson and Sinclair [8] have proved that any linear derivation on a semisimple Banach algebra is continuous. Combining this result with the Singer-Wermer theorem, one obtains that there are no nonzero linear derivations on commutative semisimple Banach algebras. Hence in case A/P is commutative, we have $D_P = 0$ as well. In other words $D(x)$ is in the intersection of all primitive ideals of A for any $x \in A$. Since the intersection of all primitive ideals is the radical, and since A is semisimple, we have $D = 0$. The proof of the theorem is complete.

As we have mentioned above, Thomas [15] has generalized the Singer-Wermer theorem by proving that any linear derivation on a commutative Banach algebra maps the algebra into its radical. This result leads to the question of whether Theorem 2 can be proved without any continuity assumption. We do not know if the answer to this question is affirmative. However, in a special case, when a Banach algebra is semisimple, one can prove the following result.

Theorem 3. *Let A be a nomcommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D\colon A \to A$, such that the mapping $x \mapsto [[D(x), x], x]$ is commuting on A. In this case $D = 0$.*

Proof. The proof goes through in the same way as the proof of Theorem 2 with the only exception that at the beginning of the proof one has to use the fact that any linear Jordan derivation on a semisimple Banach algebra is continuous (see [2, Theorem 6]).

References

1. H. E. Bell and W. S. Martindale, *Centralizing mappings of semiprime rings*, Canad. Math. Bull. **30** (1987), 92–101.
2. M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), 1003–1006.
3. M. Brešar and J. Vukman, *Jordan derivations on prime rings*, Bull. Austral. Math. Soc. **37** (1988), 321–322.
4. ——, *On some additive mappings in rings with involution*, Aequationes Math. **38** (1989), 178–185.
5. ——, *On left derivations and related mappings*, Proc. Amer. Math. Soc. **110** (1990), 7–16.
6. J. Cusack, *Jordan derivations on rings*, Proc. Amer. Math. Soc. **53** (1975), 321–324.
7. I. N. Herstein, *Jordan derivations on prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.

8. B. E. Johnson, *Continuity of derivations on commutative Banach algebras*, Amer. J. Math. **91** (1969), 1–10.
9. J. Mayne, *Ideals and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. **86** (1982), 211–213; Erratum **89** (1983), 183.
10. ——, *Centralizing mappings of prime rings*, Canad. Math. Bull. **27** (1984), 122–126.
11. E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.
12. A. M. Sinclair, *Continuous derivations on Banach algebras*, Proc. Amer. Math. Soc. **20** (1969), 166–170.
13. ——, *Jordan homomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209–214.
14. I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260–264.
15. M. P. Thomas, *The image of a derivation is contained in the radical*, Ann. of Math. (2) **128** (1988), 435–460.
16. J. Vukman, *Symmetric bi-derivations on prime and semiprime rings*, Aequationes Math. **38** (1989), 245–254.
17. ——, *Commuting and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. **109** (1990), 47–52.
18. B. Yood, *Continuous homomorphisms and derivations on Banach algebras*, Contemp. Math., vol. 32, Amer. Math. Soc., Providence, RI, 1984, pp. 279–284.

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