A NOTE ON THE LARGEST REGULAR SUBALGEBRA
OF A BANACH ALGEBRA

JYUNJI INOUE AND SIN-EI TAKAHASI

(Communicated by Paul S. Muhly)

ABSTRACT. A simple proof of Albrecht's result on the existence of the largest
closed regular subalgebra of a semisimple commutative Banach algebra is given.

Albrecht [1] proved that any semisimple commutative Banach algebra with
identity has the largest closed regular subalgebra, using the theory of decompos-
able operators. In this short note, we give a simple proof of Albrecht's result
without the assumption of semisimplicity. Our proof is quite different from
Albrecht's, and based on the consideration of the hull-kernel topology of the
carrier space of the given Banach algebra.

Theorem. If $A$ is a commutative Banach algebra with identity, then there exists
a closed regular subalgebra of $A$ which contains all closed regular subalgebras
of $A$.

Our proof is essentially based on the following

Lemma. Let $X$ be a commutative Banach algebra with identity and $B$ a Banach
subalgebra of $X$. If $B$ is regular, then for any $b \in B$ the Gelfand transform
of $b$ as an element of $X$ is continuous on the carrier space $\Phi_X$ of $X$ in the
hull-kernel topology.

Proof. We can assume without loss of generality that $B$ contains the identity
of $X$. Then it is sufficient to show that the restriction map $\theta: \Phi_X \to \Phi_B$ ;
$\varphi \to \varphi|_B$ is continuous in the hull-kernel topology. To do this let $F$ be a closed
subset of $\Phi_B$ in the hull-kernel topology. Then $\{\varphi \in \Phi_X : \varphi|_F = 0\} =
\theta^{-1}(F)$. Also since $\ker F \subset \ker \theta^{-1}(F)$, it follows that $\hull(\ker \theta^{-1}(F)) \subset \{\varphi \in \Phi_X : \varphi|_F = 0\}$. Therefore $\theta^{-1}(F)$ is closed in the hull-kernel topology. In
other words, $\theta$ is continuous in this topology. □

Proof of Theorem. Let $\text{reg}(A)$ be the closed subalgebra of $A$ generated by the
class of all closed regular subalgebras of $A$. It is sufficient to show that $\text{reg}(A)$
is regular. Note that $\text{reg}(A)$ is a commutative Banach algebra with identity. If

Received by the editors November 29, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 46H05.
Key words and phrases. Commutative Banach algebra, regular, hull-kernel topology.
Supported in part by a Grant-in-Aid for Scientific Research from the Japanese Ministry of
Education.

©1992 American Mathematical Society
0002-9939/92 $1.00 + .25 per page

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

961
$b$ is an element of $A$ which belongs to a certain regular Banach subalgebra $B$ of $A$, then $B \subseteq \text{reg}(A)$. Hence by the preceding lemma the Gelfand transform of $b$ as an element of $\text{reg}(A)$ is continuous on $\Phi_{\text{reg}(A)}$ in the hull-kernel topology. Since $\text{reg}(A)$ is generated by such $b$'s, the Gelfand transform of an arbitrary element of $\text{reg}(A)$ is also continuous on $\Phi_{\text{reg}(A)}$ in the hull-kernel topology. Consequently $\text{reg}(A)$ is regular. \(\square\)

**REFERENCES**


Department of Mathematics, Hokkaido University, Sapporo 060, Japan

Department of Basic Technology, Yamagata University, Yonezawa 992, Japan