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THE WEYL TRANSFORM AND L^p FUNCTIONS ON PHASE SPACE

BARRY SIMON

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ABSTRACT. This is primarily a negative paper showing that a bound of the form $\|W(f)\|_{\text{operator norm}} \le c\|f\|_p$ fails for the Weyl transform if $p > 2$. L^p properties of Wigner distribution functions are discussed as well as Cwikel's theorem.

Trace ideal properties of operators of the form $f(x)g(-i\nabla)$ on $L^2(\mathbb{R}^n)$ have been an important element in the study of Schrödinger operators (both scattering theory and, via the Birman-Schwinger principle, bound state problems) and Yukawa quantum field theories (see [4, Chapter 4]). The main results here are

Theorem 1. *If $f, g \in L^p(\mathbb{R}^n)$, $2 \le p < \infty$, then $f(x)g(-i\nabla) \in \mathscr{I}_p$ and $\|f(x)g(-i\nabla)\|_p \le \|f\|p\|g\|_p$.*

Theorem 2 (Cwikel). *If $f \in L^p(\mathbb{R}^n)$ and $g \in L^p_w(\mathbb{R}^n)$, $2 < p < \infty$, then $f(x)g(-i\nabla) \in \mathscr{I}_p^w$ and $\|f(x)g(-i\nabla)\|_{p,w} \le c_p\|f\|_p\|g\|_{p,w}$.*

It is natural to try to extend this to some nonproduct functions. Define for $F \in \mathscr{S}(\mathbb{R}^{2n})$

$$W(F) = (2\pi)^{-n} \int \widehat{F}(k, y)e^{i(kX+yP)}\, dk\, dy,$$

the Weyl quantization, and its asymmetrical form

$$A(F) = (2\pi)^{-n} \int \widehat{F}(k, y)e^{ikX}e^{iyP}\, dk\, dy.$$

Then $A(f(x)g(p)) = f(x)g(-i\nabla)$, so one might expect that Theorem A extends to a result of the form

$$\|A(F)\|_p \le c\|F\|_p \quad \text{or} \quad \|W(F)\|_p \le c\|F\|_p. \tag{1}$$

At first sight this might seem incompatible with the fact that $f, g \in L^p_w$ does not imply that $f(x)g(-i\nabla)$ is even compact (consider $f(x) = g(x) = |x|^{-n/p}$); but in fact, it is consistent, for $f \in L^p_w$ and $g \in L^p_w$ does *not* imply that $f(x)g(y)$ on $\mathbb{R}^{2n}$ is in weak L^p, e.g., $f(x) = g(x) = |x|^{-n/p}$ where $|\{(x, y) | f(x)g(y) \ge 1\}| = \infty$. But $f \in L^p$ and $g \in L^p_w$ imply that $f(x)g(y) \in L^p_w(\mathbb{R}^{2n})$ by a simple argument. Indeed, for f fixed, $f(x)g(y) \in L^p_w(\mathbb{R}^{2n})$

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for all $g \in L^p_w(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$. Thus if (1) holds, a simple consequence would be Cwikel's theorem via a standard application of weak interpolation theorems.

On the other hand, since it is natural to view the map $F \mapsto A(F)$ as an operator-valued Fourier transform [2], one would not expect Theorem A to extend to a result of the form (1). This is confirmed in the following theorem.

Theorem 1. *Let $p \geq 2$. Then (1) holds if and only if $p = 2$.*

That (1) holds if $p = 2$ follows formally from

$$\mathrm{Tr}(e^{i(kX+yP)}) = (2\pi)^{n/2}\delta(k)\delta(y) = \mathrm{Tr}(e^{ikX}e^{iyP}),$$

which yields

$$\mathrm{Tr}(W^*(F)W(F)) = \mathrm{Tr}(A^*(F)A(F)) = (2\pi)^{-n}\int |F(x\,,\,k)|^2\,dx\,dk.$$

A proof of this well-known fact [1, 3] follows from the Plancherel theorem and the explicit form of the integral kernels for W and A:

$$W(F)(x\,,\,z) = (2\pi)^{-n}\int \widehat{F}(k\,,\,z-x)e^{ikx}e^{ik(z-x)/2}\,dk\,,$$

$$A(F)(x\,,\,z) = (2\pi)^{-n}\int \widehat{F}(k\,,\,z-x)e^{ikx}\,dk.$$

So we turn to proving that (1) fails for $p > 2$. Indeed, we will even prove that

$$\|A(F)\|_\infty \leq c\|F\|_p\,, \qquad \|W(F)\|_\infty \leq c\|F\|_p$$

both fail where $\|\cdot\|_\infty$ is the operator norm.

This will follow from the simple duality argument. If we define for $\psi \in L^2(\mathbb{R}^n)$

$$\rho^A(\psi)(x\,,\,p) = (2\pi)^{-2n}\int e^{-ixk}e^{-ipy}\langle\psi|e^{ikX}e^{iyP}|\psi\rangle\,dk\,dy\,,$$

$$\rho^W(\psi)(x\,,\,p) = (2\pi)^{-2n}\int e^{-ixk}e^{-ipy}\langle\psi|e^{i(kX+yP)}|\psi\rangle\,dx\,dy$$

(with $\langle\psi|B|\psi\rangle = (\psi\,,\,B\psi)$); then

$$\langle\psi|A(F)|\psi\rangle = \int F(x\,,\,p)\rho^A(\psi)(x\,,\,p)\,dx\,dp\,,$$

$$\langle\psi|W(F)|\psi\rangle = \int F(x\,,\,p)\rho^W(\psi)(x\,,\,p)\,dx\,dp.$$

From this we conclude:

Proposition. *If $\|A(F)\|_\infty \leq c\|F\|_p$ (resp., $\|W(F)\|_\infty \leq c\|F\|_p$), then for $p' = p/p-1$ we have that $\|\rho^A(\psi)\|_{p'} \leq c$ (resp., $\|\rho^W(\psi)\|_{p'} \leq c$) for all $\psi \in L^2(\mathbb{R}^n)$.*

By straightforward calculation,

$$\rho^A(\psi)(x\,,\,p) = (2\pi)^{-n/2}e^{ipx}\psi^*(x)\hat{\psi}(p)\,, \tag{2}$$

$$\rho^W(\psi)(x\,,\,p) = (2\pi)^{-n}\int e^{ipy}\psi^*(x-\tfrac{1}{2}y)\psi(x+\tfrac{1}{2}y)\,dy. \tag{3}$$

From (2) we see that $\rho^A(\psi) \in L^q$ if and only if both ψ and $\hat{\psi}$ lie in L^q. Since ψ is arbitrary in L^2, there are ψ with $\rho^A(\psi) \in L^q$ if and only if $q = 2$; so by the proposition, $\|A(f)\|_\infty \leq c\|f\|_p$ only if $p = 2$.

The Weyl case is a little more subtle. Note first that if ψ is supported in $\{x \mid |x| \leq 1\}$, then by (3), $\rho^W(\psi)(x, p) \neq 0$ only if there is a y with $|x \pm \frac{1}{2}y| \leq 1$; so $|x| \leq \frac{1}{2}|(x + \frac{1}{2}y) + (x - \frac{1}{2}y)| \leq 1$, i.e., $\rho^W(\psi)(x, p) = 0$ if $|x| \geq 1$. Suppose that $\int |\rho^W|^q \, dx \, dp < \infty$. Then it follows, since the characteristic function of the unit ball is in $L^{q/q-1}$, that we have for any $\theta(x, p)$

$$\int \rho^W(x, p) e^{i\theta(x,p)} \, dx \in L^q(dp). \tag{4}$$

We will find ψ and θ in (4) false if $1 \leq q < 2$. Pick $\theta(x, p) = 2px$. Then by (3)

$$\begin{aligned} \int e^{i\theta(x,p)} \rho^W(x, p) \, dx &= (2\pi)^{-n} \int e^{2ip(x-y/2)} \psi^*(x - \tfrac{1}{2}y)\psi(x + \tfrac{1}{2}y) \, dy \, dx \\ &= (2\pi)^{-n} \int e^{2ipu} \psi(u) \psi^*(z) \, dy \, dz \\ &= (2\pi)^{-n/2} \overline{\hat{\psi}(0)} \hat{\psi}(2p). \end{aligned}$$

Now take

$$\psi(x) = \begin{cases} |x|^{-\alpha}, & |x| < 1, \\ 0, & |x| < 1 \end{cases}$$

with $2\alpha < n$. Then, $\psi(0) \neq 0$ and $\hat{\psi}(p) \sim |p|^{-(n-\alpha)}$ for p large; so $\rho^W \notin L^q$ if $q(n - \alpha) < n$. Since α can be arbitrarily closer to $n/2$, q can be chosen anywhere in $[1, 2)$. This concludes the proof of Theorem 1.

Along the way, we proved the following of independent interest:

Theorem 2. *In general, $\rho^A(\psi)$ may not lie in any L^p, $p \neq 2$. In general, $\rho^W(\psi)$ may not lie in L^p, $1 \leq p < 2$.*

It is easy to show $\rho^W \in L^\infty$ so in L^p, $2 \leq p \leq \infty$. Since ρ^W is a "density", $\int |\rho^W| \, dx \, dp = \infty$ is notable!

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Division of Physics, Mathematics and Astronomy, California Institute of Technology, 253-37, Pasadena, California 91125

E-mail address: bsimon@caltech.edu