ON A PROBLEM OF NIRENBERG
CONCERNING EXPANDING MAPS IN HILBERT SPACE

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ABSTRACT. Let $H$ be a Hilbert space and $f : H \to H$ a continuous map which is expanding (i.e., $\|f(x) - f(y)\| \geq \|x - y\|$ for all $x, y \in H$) and such that $f(H)$ has nonempty interior. Are these conditions sufficient to ensure that $f$ is onto? This question was stated by Nirenberg in 1974. In this paper we give a partial negative answer to this problem; namely, we present an example of a map $F : H \to H$ which is not onto, continuous, $F(H)$ has nonempty interior, and for every $x, y \in H$ there is $n_0 \in \mathbb{N}$ (depending on $x$ and $y$) such that for every $n \geq n_0$

$$\|F^n(x) - F^n(y)\| \geq c^{n-m}\|x - y\|$$

where $F^n$ is the $n$th iterate of the map $F$, $c$ is a constant greater than 2, and $m$ is an integer depending on $x$ and $y$. Our example satisfies $\|F(x)\| = c\|x\|$ for all $x \in H$.

We show that no map with the above properties exists in the finite-dimensional case.

1. INTRODUCTION

In 1974 Nirenberg [9] stated the following problem:

$$(P_1)$$ Let $H$ be a Hilbert space and let $f : H \to H$ be a continuous map that is expanding and whose range contains an open set. Does $f$ map $H$ onto $H$?

This question could be generalized to the case (in this paper called $(P_2)$) when the spaces considered are Banach spaces $X, Y$.

There are several partial positive answers to $(P_1)$ and $(P_2)$ in the following cases:

(a) $X$ is finite dimensional [1, 2],
(b) $f = I - C$ where $C$ is compact or a contraction or more generally a $k$-set-contraction [6, 10],
(c) $f$ strongly monotone, i.e., there exists $s > 0$ such that [3, 7]

$$\text{Re}(f(x) - f(y), x - y) \geq s\|x - y\|^2$$

for all $x, y \in X$.

In [4] Chang and Shujie proved the surjectivity of the map $f : X \to Y$ ($X, Y$ Banach spaces) under the additional assumptions that $Y$ is reflexive, $f$ is
Fréchet-differentiable, and
\[
limit_{x \to x_0} \sup \|f'(x) - f'(x_0)\| < 1 \quad \text{for all } x_0 \in X.
\]

Seven years ago Morel and Steinlein [8] gave a beautiful counterexample to (P2) in the case when \(f\) acts in the Banach space \(L^1(N)\).

In this paper we suggest a negative answer to (P1); namely, we present an example of a map \(F : H \to H\) which is not onto, continuous, \(F(H)\) has nonempty interior, and for every \(x, y \in H\) there is \(n_0 \in \mathbb{N}\) (depending on \(x\) and \(y\)) such that for every \(n \geq n_0\)
\[
\|F^n(x) - F^n(y)\| \geq c^{n-m}\|x - y\|,
\]
where \(F^n\) is the \(n\)th iterate of \(F\), \(c\) is a constant greater than 2, and \(m\) is an integer depending on \(x\) and \(y\). This condition means that the distance between any two trajectories of the discrete dynamical system \(F : H \to H\) tends to infinity in an exponential way.

2. The example

We start by constructing a map \(f : L^2(N) \to L^2(N)\) with the following properties:

(a) \(f\) is continuous,
(b) \(B(0, 1) \subset f(L^2(N))\) where \(B(0, 1)\) is the unit ball in \(L^2(N)\),
(c) \(f(L^2(N)) \neq L^2(N)\),
(d) \(f\) is an injection.

Then we define a map \(F\) by \(F(x) := cf(x)\). Taking into account the properties of \(f\) we show that \(F\) satisfies the required assumptions.

To define \(f\) we first introduce a continuous function \(\psi : R^+ \to R^+\) such that
\[
\psi(t) := \begin{cases} 
1 & \text{for all } t \text{ so that } t \leq 1 \text{ and } 2 \leq t, \\
\alpha t & \text{for } 1 < t < 2, \\
1 & \text{for } t > 2,
\end{cases}
\]
where \(\alpha\) is a fixed number which satisfies \(0 < \alpha < 1\).

Now for every \(x \in L^2(N)\) let \(n_x\) denote the minimal natural number such that
\[
\left(\sum_{i=1}^{n_x} x_i^2\right)^{1/2} \leq \psi(\|x\|) \leq \left(\sum_{i=1}^{n_x+1} x_i^2\right)^{1/2}.
\]
(We allow \(n_x = 0\) and then the left side of the above inequality is 0.) We set
\[
f(x) := \begin{cases} 
x & \text{for all } x \text{ such that } \|x\| \leq 1 \text{ or } 2 \leq \|x\|, \\
(x_1, x_2, \ldots, x_{n_x}, \alpha_x x_{n_x+1}, \sqrt{1 - \alpha_x^2 x_{n_x+1}^2}, x_{n_x+2}, x_{n_x+3}, \ldots) & \text{for } 1 < \|x\| < 2,
\end{cases}
\]
where \(\alpha_x\) satisfies
\[
\left(\sum_{i=1}^{n_x} x_i^2 + \alpha_x^2 x_{n_x+1}^2\right)^{1/2} = \psi(\|x\|).
\]
(Of course 0 \(\leq \alpha_x < 1\); if \(x_{n_x+1} = 0\) then \(\alpha_x := 0\).)
The continuity of \( f \) and properties (b) and (c) are easy to prove. So we must only prove (d).

Before passing to the proof we make the obvious observation that

\[
\|f(x)\| = \|x\| \quad \text{for every } x \in L^2(\mathbb{N}).
\]

Taking into account this observation we show (d).

**Lemma.** Let \( x, y \in L^2(\mathbb{N}) \) and \( f(x) = f(y) \). Then \( x = y \).

**Proof.** By definition of \( f \) and (2) it is sufficient to consider the case when \( 1 < \|x\| < 2 \) and \( 1 < \|y\| < 2 \). By (2) we see immediately that \( \psi(\|x\|) = \psi(\|y\|) \), and from (1) and the fact that \( f(x) = f(y) \) it follows that \( n_x = n_y \) and, consequently, \( x_i = y_i \) for both \( i = 1, 2, \ldots, n_x \) and \( i = n_x + 2, n_x + 3, \ldots \).

Since \( \|x\| = \|y\| \) we conclude that \( |x_{n+1}^x| = |y_{n+1}^y| \) and since

\[
\alpha_x x_{n+1}^x = \alpha_y y_{n+1}^y, \quad \sqrt{1 - \alpha_x^2 x_{n+1}^x} = \sqrt{1 - \alpha_y^2 y_{n+1}^y}
\]

where \( \alpha_x \geq 0 \), we see that \( x_{n+1}^x = y_{n+1}^y \), which finishes the proof.

Now we define \( F(x) := cf(x), \ c > 2 \). We show the following

**Theorem.** The map \( F \) has the following properties:

- \( F \) is continuous,
- \( F(L^2(\mathbb{N})) \) has nonempty interior,
- \( F \) is not onto,
- for arbitrary \( x, y \in H \) there is \( n_0 \in \mathbb{N} \) (depending on \( x \) and \( y \)) such that for every \( n \geq n_0 \)

\[
\|F^n(x) - F^n(y)\| \geq c^n - m\|x - y\|
\]

where \( F^n \) is the \( n \)th iterate of \( F \), \( c \) is a constant greater than 2, and \( m \) is an integer depending on \( x \) and \( y \).

**Proof.** Properties (a_1), (b_1), (c_1) are easy to prove. We show (d_1).

By definition of \( f \) and (2), for every \( x \in L^2(\mathbb{N}) \)

\[
\|F^n(x)\| = c^n\|x\|,
\]

and there is some integer \( p \) depending on \( x \) (we choose the smallest one) such that

\[
F^n(x) = c^{n-p} F^p(x) \quad \text{for } n \geq p.
\]

Now consider the expression \( \|F^n(x) - F^n(y)\| \). By (5),

\[
\|F^n(x) - F^n(y)\| = \|c^{n-p} F^p(x) - c^{n-k} F^k(y)\| = c^{n-p} \|F^p(x) - c^{p-k} F^k(y)\|
\]

\( (k \text{ corresponds to } y \text{ according to (5)}) \), and since

\[
c^{p-k} F^k(y) = F^p(y)
\]

(without loss of generality we can assume that \( p \geq k \)) we have

\[
\|F^p(x) - c^{p-k} F^k(y)\| = \|F^p(x) - F^p(y)\| > 0 \quad \text{for } x \neq y.
\]
because \( f \), and hence \( F \), is an injection. Finally, since \( c > 2 \) there is \( n_0 \) such that for every \( n \geq n_0 \)

\[
\|F^n(x) - F^n(y)\| \geq c^{n-p}\|x - y\|
\]

and \( m := \max\{k, p\} = p \). Thus, the proof of (d1) is finished.

**Proposition.** There is no map \( F_1 \) with properties (a1), (b1), (c1), (d1), and (e1) \( \|F_1(x)\| = c\|x\| \) in the finite-dimensional case.

**Proof.** Assume that \( F_1: \mathbb{R}^n \to \mathbb{R}^n \) is such a map. Then, by (c1) and (e1) there is \( 0 \neq x_0 \notin F_1(\mathbb{R}^n) \). From (e1) it follows that \( F_1 \) maps spheres (centered at 0) into spheres, in particular it maps the sphere \( \mathcal{S} \) with radius \( \|x_0\|/c \) into the sphere with radius \( \|x_0\|/c \). By (a1) and (d1) \( F_1|_{\mathcal{S}} \) is continuous injection and because each sphere in a finite-dimensional space is compact, \( F_1|_{\mathcal{S}} \) is a homeomorphism onto a compact proper subset of the other sphere. But this contradicts the well-known theorem stating that the necessary condition for a compact set in \( \mathbb{R}^n \) to be homeomorphic to a sphere in \( \mathbb{R}^n \) is that its complement has exactly two connected components [5].

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**References**