THE $p$-PERIODICITY OF THE MAPPING CLASS GROUP
AND THE ESTIMATE OF ITS $p$-PERIOD

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Abstract. We determine completely the primes $p$ for which the Farrell-Tate cohomology of the mapping class group $\Gamma_g$ is $p$-periodic. We also estimate the $p$-period of a $p$-periodic $\Gamma_g$.

For $\Gamma$ any group of finite virtual cohomological dimension (vcd) and a prime $p$, we say that the group $\Gamma$ has $p$-periodic cohomology if there exists a positive integer $d$ such that the Farrell-Tate cohomology groups $\tilde{H}^i(\Gamma; M)$ and $\tilde{H}^{i+d}(\Gamma; M)$ have naturally isomorphic $p$-primary components for all $i \in \mathbb{Z}$ and $\mathbb{Z}\Gamma$-modules $M$. The $p$-period of $\Gamma$ is defined as the least value of $d$ [B1].

Recall that the mapping class group $\Gamma_g$ is defined to be the group of path components of orientation-preserving homeomorphisms of the orientable closed surface $S_g$ of genus $g$. We always assume $g > 1$. It is well known that the mapping class group $\Gamma_g$ is of finite vcd and the $\text{vcd}(\Gamma_g) = 4g - 5$ [H].

In this paper, we determine completely the primes $p$ for which $\Gamma_g$ is $p$-periodic. Furthermore, we estimate the $p$-period of a $p$-periodic $\Gamma_g$ by using the $p$-period of a metacyclic subgroup of $\Gamma_g$ as a lower bound and a homogeneous Chern class polynomial of the canonical homology representation $\Gamma_g \rightarrow \text{GL}(2g, \mathbb{Z})$ as an upper bound. The main results are as follows:

Theorem 1. (a) The mapping class group $\Gamma_g$ is never 2-periodic.

(b) The mapping class group $\Gamma_{kp+i}$ is always $p$-periodic when $i \not\equiv 1 \pmod{p}$ where $p$ is an odd prime and $k \geq 0$.

(c) The mapping class group $\Gamma_{kp+1}$ is $p$-periodic if and only if the interval $[(2k + 3)/p, (2k + 2)/(p - 1)]$ does not contain an integer and $k \not\equiv 0, -1 \pmod{p}$ where $p$ is an odd prime. In particular, $\Gamma_{kp+1}$ can be $p$-periodic only when $k \leq (p^2 - 5)/2$.

Theorem 2. If $k \not\equiv 0 \pmod{p}$ and $p > 2$, then $\Gamma_{(p-1)(kp-k-2)/2}$ is $p$-periodic and the $p$-period of $\Gamma_{(p-1)(kp-k-2)/2}$ is a multiple of $2(p-1)$. Moreover, if $k < (p-1)/2$, the $p$-period of $\Gamma_{(p-1)(kp-k-2)/2}$ equals $2(p-1)$.

Theorem 3. If $3 \leq d$ and $d$ divides $p-1$, then $\Gamma_{(p-1)(d-2)/2}$ is $p$-periodic and the $p$-period of $\Gamma_{(p-1)(d-2)/2}$ is a multiple of $2d$.

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Theorem 4. Let \( 2g - 2 = mp - i \) where \( 0 \leq i \leq p - 1 \), \( p \) is an odd prime, and \( p^{r-1} \leq m \leq p^r \). Assume that \( \Gamma_g \) is \( p \)-periodic. Then

(a) If \( [2g/(p - 1)] < p^r \), then the \( p \)-period of \( \Gamma_g \) divides \( 2p^{r-1}(p - 1) \).

(b) If \( [2g/(p - 1)] \geq p^r \), then the \( p \)-period of \( \Gamma_g \) divides \( 2p^r(p - 1)^2 \).

The remainder of this paper is organized as follows: In the first section we prove Theorem 1. In §2 we give a lower bound for the \( p \)-period of \( \Gamma_g \) for \( g = k(p - 1) \) and thus prove Theorem 3. In §3 we give an upper bound for the \( p \)-period of \( \Gamma_g \) for \( g = k(p - 1) \) and thus prove Theorem 2. In §4 we give an upper bound for the \( p \)-period of \( \Gamma_g \) when \( \Gamma_g \) is any \( p \)-periodic mapping class group and thus prove Theorem 4.

1. The \( p \)-Periodicity of the Mapping Class Group \( \Gamma_g \)

For a group \( \Gamma \) of finite vcd, recall that \( \Gamma \) is \( p \)-periodic if and only if \( \Gamma \) does not contain \( Z/p \times Z/p \) [B1]. Furthermore, the positive solution of the Nielsen conjecture by Kerckhoff [K] implies that the finite group \( F \) is a subgroup of \( \Gamma_g \) if and only if \( F \) is isomorphic to a subgroup of \( \text{Homeo}^+(S_g) \), the group of orientation preserving homeomorphisms of \( S_g \).

Proposition 1.1. The finite group \( F \) is isomorphic to a subgroup of \( \text{Homeo}^+(S_g) \) with branching data \((h; n_1 \cdots n_b)\) if and only if \( F \) satisfies the following conditions:

1. \( F = \langle a_1, \ldots, a_h; b_1, \ldots, b_h; c_1, \ldots, c_b \rangle \);
2. \( \prod_{1 \leq i \leq h} [a_i, b_i] \prod_{i < j \leq b} c_j = 1 \);
3. \( \text{Order}(c_i) = n_i \);
4. \( \text{Riemann-Hurwitz equation} \ 2g - 2 = |F|(2h - 2) + |F| \sum_{1 \leq i \leq b}(1 - 1/n_i) \).

Proof. See [T].

Lemma 1.2. Let \( G \) be a finite subgroup of \( \text{Homeo}^+(S_g) \). Then \( G \) is also isomorphic to a subgroup of \( \text{Homeo}^+(S_{g+k|G|}) \). Here \( k \) is a nonnegative integer.

Proof. It follows immediately by Proposition 1.1.

Proof of Theorem 1. (a) We only need to show \( \Gamma_i \supseteq Z/2 \times Z/2 \) for \( 2 \leq i \leq 5 \) by Lemma 1.2. In fact, write \( Z/2 \times Z/2 = \langle x, y | x^2 = y^2 = 1, xy = yx \rangle \) in the following forms:

(i) when \( g = 2 \), \( Z/2 \times Z/2 = \langle x, x, x, x, xy \rangle, h = 0, b = 5 \);
(ii) when \( g = 3 \), \( Z/2 \times Z/2 = \langle x; y; x, x \rangle, h = 1, b = 2 \);
(iii) when \( g = 4 \), \( Z/2 \times Z/2 = \langle x; y; x, y, xy \rangle, h = 1, b = 3 \);
(iv) when \( g = 5 \), \( Z/2 \times Z/2 = \langle x; y; x, y, x, y \rangle, h = 1, b = 4 \).

It is straightforward to check by Proposition 1.1 that (i) \( \Gamma_2 \supseteq Z/2 \times Z/2 \) with branching data \((0; 2, 2, 2, 2)\), (ii) \( \Gamma_3 \supseteq Z/2 \times Z/2 \) with branching data \((1; 2, 2)\), (iii) \( \Gamma_4 \supseteq Z/2 \times Z/2 \) with branching data \((1; 2, 2, 2)\), and (iv) \( \Gamma_5 \supseteq Z/2 \times Z/2 \) with branching data \((1; 2, 2, 2)\). We have proved that \( \Gamma_2 \) is never 2-periodic.

In the rest of this paper, \( p \) is an odd prime.

(b) \( \Gamma_{kp+i} \) is \( p \)-periodic for \( i \neq 1 \). Otherwise, we have \( \Gamma_{kp+i} \supseteq Z/p \times Z/p \).

The Riemann-Hurwitz formula holds: \( 2(kp+i-2) = p^2(2h-2)+p^2(1-1/p)b \), i.e., \( 2k + (2i-2)/p = p(2h-2) + (p-1)b \) implies \( 2i-2 = 0 \mod(p) \), forcing \( i = 1 \mod(p) \). This is a contradiction.
(c) Claim 1. If \( k = 0 \) or \(-1 \mod(p)\) or the interval \([(2k + 3)/p, (2k + 2)/(p - 1)]\) contains an integer, then \( \Gamma_{kp+1} \supseteq \mathbb{Z}/p \otimes \mathbb{Z}/p \).

Case 1. \( k = 0 \mod(p) \). Suppose \( k = np \), where \( n \) is a nonnegative integer. We show a \( \mathbb{Z}/p \times \mathbb{Z}/p \) free action on \( S_{kp+1} \) by Proposition 1.1. In fact, write \( \mathbb{Z}/p \times \mathbb{Z}/p = \langle x, y | x^p = y^p = 1 \rangle \), \( xy = yx \) \( = (x_1, x_2, \ldots, x_{n+1}, \ y_1, y_2, \ldots, y_{n+1}) \), \( h = 0, b = n + 1 \), where \( x_i = x \) and \( y_i = y, 1 \leq i \leq n + 1 \). Notice that the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2(n+1)-2) \) holds.

Case 2. \( k = -1 \mod(p) \). We show that there is a \( \mathbb{Z}/p \times \mathbb{Z}/p \) action with two singular points on \( S_{kp+1} \). Write

\[
\mathbb{Z}/p \times \mathbb{Z}/p = \langle x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n; x, x^{-1} \rangle,
\]

\( h = n, b = 2 \), where \( n = (k + 1)/p \geq 1, x_i = x, y_i = y, 1 \leq i \leq n \). Notice that the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2(n+1)-2) + p^2(1-1/p) \) holds.

Case 3. The interval \([(2k + 3)/p, (2k + 2)/(p - 1)]\) contains an integer \( n \). We show that there is a \( \mathbb{Z}/p \times \mathbb{Z}/p \) action on \( S_{kp+1} \) with \( t = np - 2k \) singular points. Note \( t = np - 2k \geq 3 \).

Let \( h = k + 1 - (n(p-1))/2 \), then \( h \geq k + 1 - (2k+2)/2 = 0 \). Write \( \mathbb{Z}/p \times \mathbb{Z}/p = \langle x_1, x_2, \ldots, x_h; y_1, y_2, \ldots, y_h; y, x_1, x_2, \ldots, x_{t-2}, (\Pi_{1 \leq i \leq t-2} x_i)^{-1} y^{-1} \rangle \).

Here \( x_i = x_j = x \) and \( y_i = y, 1 \leq i \leq h, 1 \leq j \leq t - 2 \). Note that the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2h - 2) + p^2(1-1/p)t \) holds, i.e., \( 2kp = p^2(2k - n(p - 1)) + p(p - 1)(np - 2k) = 2kp^2 - np^2(p - 1) + np^2(p - 1) - 2kp(p - 1) \).

Claim 2. Conversely, if \( \Gamma_{kp+1} \) is not \( p \)-periodic, then \( k = 0 \mod(p), k = -1 \mod(p), \) or the interval \([(2k + 3)/p, (2k + 2)/(p - 1)]\) contains an integer.

Let \( \Gamma_{kp+1} \supseteq \mathbb{Z}/p \times \mathbb{Z}/p \), i.e., there exists a \( \mathbb{Z}/p \times \mathbb{Z}/p \) action on \( S_{kp+1} \).

Case 1. The \( \mathbb{Z}/p \times \mathbb{Z}/p \) acts freely on \( S_{kp+1} \). Then the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2h - 2) \) implies \( k = p(h - 1), \) i.e., \( k = 0 \mod(p) \).

Case 2. The \( \mathbb{Z}/p \times \mathbb{Z}/p \) acts on \( S_{kp+1} \) with two singular points. Then the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2h - 2) + p^2(1-1/p) \) implies \( k = p(h - 1) + p - 1 \), i.e., \( k = -1 \mod(p) \).

Case 3. The \( \mathbb{Z}/p \times \mathbb{Z}/p \) acts on \( S_{kp+1} \) with more than three singular points. Then the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2h - 2) + p^2(1-1/p)t \) implies \( k = p(h - 1) + p - 1 \), i.e., \( k = -1 \mod(p) \).

We have proved Theorem 1.
2. The metacyclic subgroups of the mapping class group

Let \( M_{p,k} = \langle a, b \mid a^{k(p-1)} = 1, b^p = 1, aba^{-1} = b^s \rangle \) and \( N_{p,d} = \langle a, b \mid a^d = 1, b^p = 1, aba^{-1} = b^t \rangle \) denote metacyclic groups, where \( p \) is an odd prime, \( k \) is a positive integer, \( d \geq 3 \) and divides \( p - 1 \), \( s \) and \( t \) are mod(\( p \)) integers such that the order of \( s \) in the multiple group \( (\mathbb{Z}/p\mathbb{Z})^* \) is equal to \( p - 1 \), and the order of \( t \) in the multiple group \( (\mathbb{Z}/p\mathbb{Z})^* \) is equal to \( d \). Then the order \( |M_{p,k}| = kp(p-1) \) and \( |N_{p,d}| = pd \).

Lemma 2.1. (a) \( \Gamma_{(p-1)(kp-2-k)/2} \supset M_{p,k} \) except for \( k = 1, p = 3 \).
(b) \( \Gamma_{(p-1)(d-2)/2} \supset N_{p,d} \).

Proof. We apply Lemma 1.1 to show \( \text{Homeo}^+(S_{(p-1)(kp-2-k)/2}) \supset M_{p,k} \) and \( \text{Homeo}^+(S_{(p-1)(d-2)/2}) \supset N_{p,d} \). Write the groups \( M_{p,k} = \langle b^{-1}, a, a^{-1}b \rangle \) (\( h = 0, b = 3 \)) and \( N_{p,d} = \langle b^{-1}, a, a^{-1}b \rangle \) (\( h = 0, b = 3 \)). Note the order \( O(a^{-1}b) = k(p-1) \) in \( M_{p,k} \) and the order \( O(a^{-1}b) = d \) in \( N_{p,d} \). It is direct to check that the Riemann-Hurwitz formula holds for both groups \( M_{p,k} \) and \( N_{p,d} \):

\[
2(p-1)(kp-2-k)/2 - 2 = kp(p-1)(2(0)-2) + kp(p-1)(1-1/p)1
+ kp(p-1)(1-1/[k(p-1)])/2,
2(p-1)(d-1)/2 - 2 = pd(2(0)-2) + pd(1-1/p) + pd(1-1/d)2.
\]

Lemma 2.2. The finite group \( M_{p,k} \) (resp. \( N_{p,d} \)) is \( p \)-periodic with the \( p \)-period \( 2(p-1) \) (resp. \( 2d \)) for \( k \not\equiv 0 \mod(p) \).

Proof. The order \( |M_{p,k}| = kp(p-1) \) (resp. \( |N_{p,d}| = pd \)) implies that the group \( M_{p,k} \) (resp. \( N_{p,d} \)) does not contain \( Z/p \times Z/p \) except \( k = 0 \mod(p) \), i.e., \( M_{p,k} \) (resp. \( N_{p,d} \)) is \( p \)-periodic. We will compute the \( p \)-period of \( M_{p,k} \) (resp. \( N_{p,d} \)) by using a result of Swan [S] that states that the \( p \)-period of a \( p \)-periodic finite group equals \( 2|N(S_p)/C(S_p)| \) for an odd prime \( p \), where \( N(-) \) and \( C(-) \) denote the normalizer and centralizer and \( S_p \) is a \( p \)-Sylow subgroup of \( G \). In fact, the order \( |N(S_p)/C(S_p)| = |N(Z/p)/C(Z/p)| = kp(p-1)/kp = p-1 \) in \( M_{p,k} \) and the order \( |N(Z/p)/C(Z/p)| = pd/p = d \) in \( N_{p,d} \). This completes the proof of Lemma 2.2.

The \( p \)-periodicity of \( \Gamma_{(p-1)(kp-2-k)/2} \) and \( \Gamma_{(p-1)(d-2)/2} \) are clear by Theorem 1(b). So a lower bound of the \( p \)-period of certain mapping class group is obtained by combining Lemmas 2.1 and 2.2.

We have thus proved Theorem 3 and the following lemma.

Lemma 2.3. The \( p \)-period of \( \Gamma_{(p-1)(kp-2-k)/2} \) is a multiple of \( 2(p-1) \).

3. The Chern classes of the canonical homology representation of the mapping class group

Recall that for a complex representation \( f: \Gamma \to \text{GL}(k, \mathbb{C}) \) of the discrete group \( \Gamma \) the Chern classes \( c_i(f) \in H^{2i}(\Gamma; \mathbb{C}) \) are defined as Chern classes of the flat \( C^k \)-bundle over \( K(\Gamma, 1) \) classified by \( Bf: K(\Gamma, 1) \to B\text{GL}(k, \mathbb{C}) \). Let \( Q \) be a subring in \( \mathbb{C} \). If \( f: \Gamma \to \text{GL}(k, Q) \) is a representation over \( Q \), we will write \( c_i(f) \) for the \( i \)th Chern class of the associated complex representation \( \Gamma \to \text{GL}(k, \mathbb{Q}) \to \text{GL}(k, \mathbb{C}) \).
It is well known that over \( Q \) the group \( \mathbb{Z}/n\mathbb{Z} \) has a unique faithful irreducible representation \( \sigma_n : \mathbb{Z}/n\mathbb{Z} \to \text{GL}(\varphi(n), Q) \), where \( \varphi(n) \) is the Euler function. Glover and Mislin showed a result in [GM].

**Proposition 3.1** (Glover and Mislin). Let \( r : \mathbb{Z}/p^a \to \text{GL}(k, Q) \) be a \( Q \) representation. Suppose that in the decomposition of \( r \) into \( Q \) irreducible representation \( \sigma_{p^a} \) occurs with multiplicity \( m \), where \( m \) is not divisible by \( p \). Then for every \( j > 0 \), \( (C_{p^j}(\varphi(r))^j \in H^{2j}(\mathbb{Z}/p^a, \mathbb{Z}) \) has order \( p^a \).

Let \( \mu : \Gamma_g \to \text{Sp}(2g, \mathbb{Z}) \) be the map obtained by allowing a homeomorphism \( h \) of \( S_g \) to act on \( H_1(S_g; \mathbb{Z}) \) and let \( i : \text{Sp}(2g, \mathbb{Z}) \to \text{GL}(2g, Q) \) be the canonical inclusion. Then \( \varepsilon = i\mu : \Gamma_g \to \text{GL}(2g, Q) \) is a representation over \( Q \). If \( p : \mathbb{Z}/p \to \Gamma_g \to \text{GL}(2g, Q) \) is the composite of inclusion and \( \varepsilon \), and since \( \rho \) is faithful, \( \chi_{\rho} = m_{\rho} \chi_{\text{tr}} + n_{\rho} \chi_{\sigma} \), where \( \chi \) stands for the character of the representation, \( \text{tr} \) denotes the trivial representation, \( \sigma \) is the unique irreducible representation of \( \mathbb{Z}/p \), and the integers \( m_{\rho} \) and \( n_{\rho} \) depend on \( \rho \) [Se].

**Proposition 3.2.** Suppose \( \Gamma_g \) is \( p \)-periodic for an odd prime \( p \), \( p = ei : \mathbb{Z}/p \to \text{GL}(k, Q) \) is a representation of \( \mathbb{Z}/p \) over \( Q \) for any inclusion \( i : \mathbb{Z}/p \to \Gamma_g \), and \( \chi_{\rho} = m_{\rho} \chi_{\text{tr}} + n_{\rho} \chi_{\sigma} \). If \( n_{\rho} \) is not divisible by \( p \) where \( i \) ranges over all inclusions, then \( \Gamma_g \) has \( p \)-period \( m_{\rho} \), dividing \( 2\varphi(p) = 2(p - 1) \).

**Proof.** It is well known that \( \Gamma_g \) is \( p \)-periodic (\( p > 2 \)) if and only if every \( p \)-Sylow subgroup \( S_p \) of \( \Gamma_g \) is cyclic. We need to show that there exists an element \( a \in \tilde{H}^{2\varphi(p)}(\Gamma_g; \mathbb{Z}) \) such that \( \text{Res}(a) \in \tilde{H}^{2\varphi(p)}(\mathbb{Z}/p; \mathbb{Z}) \) is nontrivial for every \( \mathbb{Z}/p \) inclusion by the Brown-Venkov theorem.

Let \( g^* : H^{2\varphi(p)}(\Gamma_g; \mathbb{Z}) \to \tilde{H}^{2\varphi(p)}(\Gamma_g; \mathbb{Z}) \) be the canonical map from the ordinary cohomology to the Farrell-Tate cohomology. The following diagram is commutative:

\[
\begin{array}{ccc}
H^{2\varphi(p)}(\Gamma_g; \mathbb{Z}) & \xrightarrow{\text{Res}} & \tilde{H}^{2\varphi(p)}(\mathbb{Z}/p; \mathbb{Z}) \\
\uparrow g^* & & \uparrow g^* \\
\tilde{H}^{2\varphi(p)}(\Gamma_g; \mathbb{Z}) & \xrightarrow{\text{Res}} & H^{2\varphi(p)}(\mathbb{Z}/p; \mathbb{Z})
\end{array}
\]

Let \( a = g^*[c_{p-1}(\varepsilon)] \in \tilde{H}^{2\varphi(p)}(\Gamma_g; \mathbb{Z}) \). Then \( \text{Res}(a) = \text{Res} g^*[c_{p-1}(\varepsilon)] = g^* \text{Res}[c_{p-1}(\varepsilon)] = g^*[c_{p-1}(\rho)] \) has nontrivial order in \( \tilde{H}^{2\varphi(p)}(\mathbb{Z}/p; \mathbb{Z}) \) for every \( \mathbb{Z}/p \) inclusion since the irreducible representation \( \sigma \) occurs with multiplicity \( n \) that is not divisible by \( p \), i.e., the cup-product with \( g^*[c_{p-1}(\varepsilon)] \) gives an isomorphism for all integers \( i \) and a \( \Gamma_g \)-module \( Z : \tilde{H}^i(\Gamma_g; \mathbb{Z})(p) \to \tilde{H}^{2\varphi(p)+i}(\Gamma_g; \mathbb{Z})(p) \).

Next, we give an upper bound of the \( p \)-period of \( \Gamma_{(p-1)(kp-2-k)/2} \) when \( k < (p - 1)/2 \). Consider any inclusion \( i : \mathbb{Z}/p \to \Gamma_{(p-1)(kp-2-k)/2} \). The Riemann-Hurwitz formula \( 2(p - 1)(kp - 2 - k)/2 - 2 = p(2h - 2) + (p - 1)t \) implies \( t = kp - k - 2ph/(p - 1) = k(p - 1) - sp \), where \( s = 2h/(p - 1) \) must be an integer. The Lefschetz-Hopf trace formula also tells us \( \chi_{\rho}(T) = 2 - t = 2 - k(p - 1) + sp \), where \( T \) is a generator of \( \mathbb{Z}/p \) and \( \rho = ei : \mathbb{Z}/p \to \text{GL}((p-1)(kp-2-k), Q) \).

Since \( \chi_{\text{tr}}(1) = 1 \), \( \chi_{\text{tr}}(T) = 1 \), \( \chi_{\sigma}(1) = p - 1 \), and \( \chi_{\sigma}(T) = -1 \), we have \( \chi_{\rho}(1) = m + n(p - 1) = (p - 1)(kp - k - 2) \) and \( \chi_{\rho}(T) = m - n = 2 - k(p - 1) + sp \).
Then $np = (p - 1)(kp - k - 2) - 2 + k(p - 1) - sp = (kp - k - 2 - s)p$ implies $n = kp - k - 2 - s$. Notice $t \geq 0$ implies $0 \leq s \leq k - 1$. If $k < (p - 1)/2$, then $(k - 1)p < kp - k - 1 - 2 \leq kp - k - s - 2 = n < kp$. So $n$ is not divisible by $p$. By Proposition 3.2 we conclude that the $p$-period of $\Gamma_{(p-1)(kp-k)/2}$ divides $2(p - 1)$ if $0 < k < (p - 1)/2$.

We have now proved Theorem 2 by combining the results of these two sections.

4. AN UPPER BOUND OF THE $p$-PERIOD OF A $p$-PERIODIC MAPPING CLASS GROUP $\Gamma_g$

In this section, we construct a homogeneous Chern class polynomial of the canonical homology representation $\Gamma_g \to \text{GL}(2g, \mathbb{Q})$, which depends only upon the genus $g$ and the prime $p$ so that the restrictions of the homogeneous Chern class polynomial to all $\mathbb{Z}/p$ inclusions in $\Gamma_g$ are nontrivial. Therefore an upper bound of the $p$-period of a $p$-periodic $\Gamma_g$ is obtained.

We still suppose $g > 1$ and $p$ an odd prime. Let $2g - 2 = mp - i$, where $0 \leq i \leq p - 1$. Define a fixed point number set $B_{g,p}$ as follows:

- $B_{g,p} = \{i, i + p, i + 2p, \ldots, i + ((2g/(p - 1)) - m)p\}$ if $i \neq 1$,
- $B_{g,p} = \{1 + p, 1 + 2p, \ldots, 1 + ((2g/(p - 1)) - m)p\}$ if $i = 1$.

Remark. Define $B_{g,p} = \emptyset$ if in the cases (a) $i \neq 1$ and $2p/(p - 1) < m$ and (b) $i = 1$ and $2g/(p - 1) < m + 1$.

Lemma 4.1. If $\mathbb{Z}/p = \langle x \rangle$ acts on the surface $S_g$ and $2g - 2 = mp - i$, where $0 \leq i \leq p - 1$, then the number $t$ of fixed points of $x$ belongs to $B_{g,p}$. Conversely, any number $t \in B_{g,p}$ can occur as the number of fixed points of an order $p$ homeomorphism $x$ on the surface $S_g$.

Proof. We again use Proposition 1.1. If $\mathbb{Z}/p = \langle x \rangle$ acts on the surface, the Riemann-Hurwitz formula $2g - 2 = (2h - 2)p + tp(1 - 1/p)$ implies $t = 2(g - h)/(p - 1) - 2h - 2 = n - (2g - n(p - 1)) + 2 = np - 2g + 2$. Here $g - h = n(p - 1)/2$, $n$ is an integer, and $n \leq (2g/(p - 1))$ since $h \geq 0$. Therefore, $t = -2g + 2 = i \mod(p)$ and $0 \leq t \leq ((2g/(p - 1)) - m)p + i$, i.e., $t \in B_{g,p}$. Notice, if $i = 1$ then $t \neq 1$, since the number of fixed points of $\mathbb{Z}/p$ action cannot be $1$.

Conversely, if $t \in B_{g,p}$, i.e., $t = i + kp$, where $0 \leq k \leq (2g/(p - 1)) - m$ if $i \neq 1$; $0 < k \leq (2g/(p - 1)) - m$ if $i = 1$. Let $n = k + m \leq (2g/(p - 1))$. Then $h = g - n(p - 1) = g - (k + m)(p - 1)/2 \geq 0$. We show that there exists a $\mathbb{Z}/p = \langle x \rangle$ action on the surface $S_g$ with $t$ fixed points numbered by Proposition 1.1. In fact, write

$$Z/p = \langle x \rangle = \langle x_1 \cdots x_h; x_1^{-1} \cdots x_h^{-1}, x_{h+1}^{-1}, \ldots, x_{h+t-1}^{-1}, x^{-(t-1)} \rangle$$

for $t \neq 1 \mod(p)$;

$$Z/p = \langle x \rangle = \langle x_1 \cdots x_h; x_1^{-1} \cdots x_h^{-1}, x_{h+1}^2, \ldots, x_{h+t-1}^2, x^{-t} \rangle$$

for $t = 1 \mod(p)$, where $x_j = x$, $0 < j < h + t$; it is easy to check the Riemann-Hurwitz formula $(2h - 2)p + (p - 1)t = (2g - n(p - 1) - 2)p + (p - 1)(i + kp) = mp - i = 2g - 2$ since $n = k + m$.
Lemma 4.2. Let $Z/p = \langle x \rangle$ act on the surface $S_g$ and $\rho = ei: Z_p \to \Gamma_g \to \text{GL}(2g, Q)$ be a representation for an inclusion $i: Z_p \to \Gamma_g$. Then $\rho$ is equivalent to one of the following representations as a complex representation: $\rho_k = (m+k)\sigma_p \oplus nTr$, where $2g - 2 = mp - i$, $0 \leq i \leq p - 1$. If $i \neq 0$, $0 \leq k \leq \lfloor 2g/(p-1) \rfloor - m$, $i \leq k \leq \lfloor 2g/(p-1) \rfloor - m$, $n = 2g - (m+k)(p-1)$. Conversely, any representation of $\rho_k = (m+k)\sigma_p \oplus nTr$ as above can be equivalent to $\rho = ei$ for some inclusion $i: Z/p \to \Gamma_g$.

Proof. It is clear that the character $\chi_{\rho_k}(\text{Id}) = (m+k)(p-1) + n = 2g$ and $\chi_{\rho_k}(x) = (m+k) + n = 2g - (m+k)p = 2 - i - kp$. On the other hand, by using the Lefschetz fixed point theorem, for any $\rho = ei$, $\chi_{\rho}(\text{Id}) = 2g$, $\chi_{\rho}(x) = 2 - i \in 2 - B_{g,p} = \{2 - i - kp\}$. Lemma 4.1 implies $\chi_{\rho} = \chi_{\rho_k}$ for some $k$. Conversely, by Lemma 4.1, there exists a $Z/p$ representation $\rho$ such that $\chi_{\rho} = \chi_{\rho_k}$ for every $k$. This completes the proof of Lemma 4.2.

Lemma 4.3. Let $2g - 2 = mp - i$, where $p$ is an odd prime and $0 \leq i \leq p - 1$. If $m < p^r$ then $\lfloor 2g/(p-1) \rfloor < p^{r+1} - p^r$.

Proof. We have $2g - 2 + i = mp < p^{r+1}$, i.e., $2g/(p-1) < (p^{r+1} + 2)/(p-1)$. Since $3p^{r+1} + 2 \leq p^{r+2} + p^r$ implies $p^{r+1} + 2 \leq p^{r+2} - 2p^{r+1} + p^r$, we obtain $(p^{r+1} + 2)/(p-1) \leq p^{r+1} - p^r$.

Lemma 4.4. If $1 \leq k \leq p - 1$ and $kp^r \leq n < (k+1)p^r$, then $n!/p^r!(n-p^r)! = k \mod (p)$.

Proof. Write the integer $n!/p^r!(n-p^r)! = \{(p^r + 1)/(1)\} \{(p^r + 2)/(2)\} \cdot \cdots \{(p^r + p)/(p)\} \{(p^r + p + 1)/(p + 1)\} \cdot \cdots \{(p^r + 2p)/(p^2)\} \cdot \cdots \{(p^r + p^r)/(p^r + 1)\} \cdot \cdots \{(p^r + 3p^r)/(3p^r)\} \cdot \cdots \{(kp^r)/(k - 1)p^r\} \cdot \cdots \{(n)/(n - p^r)\}$.

If $i \neq 0 \mod (p)$, then $[(p^r + i)/(i)] = 1$ in the field $F_p$. If $i = 0 \mod (p^{s-1})$ and $i \neq 0 \mod (p^s)$ ($s \leq r$), then $[(p^r + i)/(i)] = 1$ in the field $F_p$. If $i = mp^r$, $m = 0, 1, 2, \ldots, k - 1$, then $[(p^r + i)/(i)] = [(m + 1)/(m)]$ in $F_p$.

So $n!/p^r!(n-p^r)! = \{(1)/(1)\} \{(2)/(1)\} \cdot \cdots \{(k)/(k - 1)\} = \{k\}$ in $F_p$, i.e., the integer $n!/p^r!(n-p^r)! = k \mod (p)$ when $kp^r \leq n < (k+1)p^r$.

Proof of Theorem 4. (a) If $2g/(p-1) < p^r$, let $\rho_k: Z/p \to \Gamma_g \to \text{GL}(2g, Q)$ be a representation equivalent to the complex $\rho_k = (m+k)\sigma_p \oplus nTr$. The total Chern class $C$ satisfies

$$C(\rho_k) = C(\sigma_p)^{m+k} = \left[1 + c_{p-1}(\sigma_p)\right]^{m+k}$$

$$= \sum_{0 \leq t \leq m+k} (m+k)!/t!(m+k-t)!(c_{p-1}(\sigma_p))^t .$$

So

$$i^*c_{\varphi(p^r)}(\varepsilon) = c_{\varphi(p^r)}(\rho_k) = (m+k)!/p^r-1!(m+k-p^r-1)!(c_{p-1}(\sigma_p))^{p^r-1} \neq 0 \mod (p) .$$
in $H^{2p^r-1}(p-1)(Z/p; Z)$ for every inclusion $i: Z/p \to \Gamma_g$ since $p^r-1 \leq m+k \leq [2g/(p-1)] < p^r$ by Lemma 4.4. Note that $\Gamma_g$ is $p$-periodic by assumption. Using an argument similar to the proof of Proposition 3.2, we obtain an upper bound $2p^r-1(p-1)$ of the $p$-period of $\Gamma_g$ in this case.

(b) If $[2g/(p-1)] \geq p^r$, consider the restriction of a homogeneous Chern class polynomial to every $Z/p$ inclusion. We have

$$i^* \{ [c_{\varphi(p^r)}(e)]^{p(p-1)} \cdot [c_{\varphi(p^{r+1})}(e)]^{p-1} \}$$

$$\quad = [c_{\varphi(p^r)}(\rho_k)]^{p(p-1)} + [c_{\varphi(p^{r+1})}(\rho_k)]^{p-1}$$

$$\quad = [(m+k)!/p^{r-1}(m+k-p^{r-1})!]^{p(p-1)} [c_{p-1}(\sigma_p)]^{p^r(p-1)}$$

$$\quad + [(m+k)!/p^{r}(m+k-p^r)!]^{p-1} [c_{p-1}(\sigma_p)]^{p^r(p-1)}$$

in $H^{2p^r(p-1)^2}(Z/p; Z)$.

If $m+k < p^r$, the second term above vanishes and the first term is nontrivial by Lemma 4.4 since $p^r-1 \leq m+k < p^r$. If $m+k \geq p^r$, then $m+k \leq [2g/(p-1)] < p^{r+1} - p^r$ by Lemma 4.3. It implies that the second term above always equals 1 mod($p$) and the first term above is 0 or 1 mod($p$). So, the element $i^* \{ [c_{\varphi(p^r)}(e)]^{p(p-1)} + [c_{\varphi(p^{r+1})}(e)]^{p-1} \} = [c_{\varphi(p^r)}(\rho_k)]^{p(p-1)} + [c_{\varphi(p^{r+1})}(\rho_k)]^{p-1}$ has order $p$ in $H^{2p^r(p-1)^2}(Z/p; Z)$ for every $Z/p$ inclusion. It follows that $2p^r(p-1)^2$ is an upper bound of the $p$-period of a periodic $\Gamma_g$, and we have completed the proof of Theorem 4.

**Remark.** The upper bound of the $p$-period of $\Gamma_g$ in Theorem 4 is a little bit rough. It can be improved by individually computing the Chern classes of the homology representation of $\Gamma_g$ with the same method.

**Example.** Consider the 3-periodic group $\Gamma_3$, $i: Z/3 \to \Gamma_3$ is an inclusion, the number of possible fixed points are 2 or 5; i.e., the associated representations are $\rho_1 = 2\sigma_3 \oplus 2Tr$ or $\rho_2 = 2\sigma_3$. But $i_1 \ast c_2(e) = c_2(\rho_1) = c_2(2\sigma_3 \oplus 2Tr) = 2c_2(\sigma_3)$, $i_1 \ast c_6(e) = c_6(\rho_1) = c_6(2\sigma_3 \oplus 2Tr) = 0$. Therefore, $i_1 \ast \{ [c_2(e)]^3 \cdot c_6(e) \} = 2[c_2(\sigma_3)]^3$ is nontrivial. Similarly, $i_2 \ast c_2(e) = c_2(\rho_2) = c_2(3\sigma_3) = 3c_2(\sigma_3) = 0 \mod(3)$, $i_2 \ast c_6(e) = c_6(\rho_2) = c_6(3\sigma_3) = [c_2(\sigma_3)]^3$. So, $i_2 \ast \{ [c_2(e)]^3 + c_6(e) \} = c_2(\sigma_3)$ is nontrivial, i.e., the element $[c_2(e)]^3 + c_6(e)$ is nontrivial when restricted to every $Z/3$ subgroup. Thus, we obtain an upper bound 12 of the 3-period of $\Gamma_3$, which is better than the upper bound 24 given by Theorem 4.

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**References**


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