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THE p-PERIODICITY OF THE MAPPING CLASS GROUP AND THE ESTIMATE OF ITS p-PERIOD

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ABSTRACT. We determine completely the primes p for which the Farrell-Tate cohomology of the mapping class group Γ_g is p-periodic. We also estimate the p-period of a p-periodic Γ_g.

For Γ any group of finite virtual cohomological dimension (vcd) and a prime p, we say that the group Γ has p-periodic cohomology if there exists a positive integer d such that the Farrell-Tate cohomology groups $\widehat{H}^i(\Gamma; M)$ and $\widehat{H}^{i+d}(\Gamma; M)$ have naturally isomorphic p-primary components for all $i \in Z$ and $Z\Gamma$-modules M. The p-period of Γ is defined as the least value of d [B1].

Recall that the mapping class group Γ_g is defined to be the group of path components of orientation-preserving homeomorphisms of the orientable closed surface S_g of genus g. We always assume $g > 1$. It is well known that the mapping class group Γ_g is of finite vcd and the $\operatorname{vcd}(\Gamma_g) = 4g - 5$ [H].

In this paper, we determine completely the primes p for which Γ_g is p-periodic. Furthermore, we estimate the p-period of a p-periodic Γ_g by using the p-period of a metacyclic subgroup of Γ_g as a lower bound and a homogeneous Chern class polynomial of the canonical homology representation $\Gamma_g \to \mathrm{GL}(2g, Z)$ as an upper bound. The main results are as follows:

Theorem 1. (a) *The mapping class group Γ_g is never 2-periodic.*

(b) *The mapping class group Γ_{kp+i} is always p-periodic when $i \neq 1 \mod(p)$ where p is an odd prime and $k \geq 0$.*

(c) *The mapping class group Γ_{kp+1} is p-periodic if and only if the interval $[(2k+3)/p,\ (2k+2)/(p-1)]$ does not contain an integer and $k \neq 0,\ -1 \mod(p)$ where p is an odd prime. In particular, Γ_{kp+1} can be p-periodic only when $k \leq (p^2-5)/2$.*

Theorem 2. *If $k \neq 0 \mod(p)$ and $p > 2$, then $\Gamma_{(p-1)(kp-k-2)/2}$ is p-periodic and the p-period of $\Gamma_{(p-1)(kp-k-2)/2}$ is a multiple of $2(p-1)$. Moreover, if $k < (p-1)/2$, the p-period of $\Gamma_{(p-1)(kp-k-2)/2}$ equals $2(p-1)$.*

Theorem 3. *If $3 \leq d$ and d divides $p-1$, then $\Gamma_{(p-1)(d-2)/2}$ is p-periodic and the p-period of $\Gamma_{(p-1)(d-2)/2}$ is a multiple of $2d$.*

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Theorem 4. *Let* $2g-2 = mp - i$ *where* $0 \le i \le p-1$, p *is an odd prime, and* $p^{r-1} \le m \le p^r$. *Assume that* Γ_g *is p-periodic. Then*

(a) *If* $[2g/(p-1)] < p^r$, *then the p-period of* Γ_g *divides* $2p^{r-1}(p-1)$.
(b) *If* $[2g/(p-1)] \ge p^r$, *then the p-period of* Γ_g *divides* $2p^r(p-1)^2$.

The remainder of this paper is organized as follows: In the first section we prove Theorem 1. In §2 we give a lower bound for the p-period of Γ_g for $g = k(p-1)$ and thus prove Theorem 3. In §3 we give an upper bound for the p-period of Γ_g for $g = k(p-1)$ and thus prove Theorem 2. In §4 we give an upper bound for the p-period of Γ_g when Γ_g is any p-periodic mapping class group and thus prove Theorem 4.

1. The p-periodicity of the mapping class group Γ_g

For a group Γ of finite vcd, recall that Γ is p-periodic if and only if Γ does not contain $Z/p \times Z/p$ [B1]. Furthermore, the positive solution of the Nielsen conjecture by Kerckhoff [K] implies that the finite group F is a subgroup of Γ_g if and only if F is isomorphic to a subgroup of $\mathrm{Homeo}^+(S_g)$, the group of orientation preserving homeomorphisms of S_g.

Proposition 1.1. *The finite group* F *is isomorphic to a subgroup of* $\mathrm{Homeo}^+(S_g)$ *with branching data* $(h; n_1 \cdots n_b)$ *if and only if* F *satisfies the following conditions:*

(1) $F = \langle a_1, \dots, a_h; b_1, \dots, b_h; c_1, \dots, c_b\rangle$;
(2) $\prod_{1\le i\le h}[a_i, b_i]\prod_{i\le j\le b} c_j = 1$;
(3) $\mathrm{Order}(c_i) = n_i$;
(4) *Riemann-Hurwitz equation* $2g-2 = |F|(2h-2)+|F|\sum_{1\le i\le b}(1-1/n_i)$.

Proof. See [T].

Lemma 1.2. *Let* G *be a finite subgroup of* $\mathrm{Homeo}^+(S_g)$. *Then* G *is also isomorphic to a subgroup of* $\mathrm{Homeo}^+(S_{g+k|G|})$. *Here* k *is a nonnegative integer.*

Proof. It follows immediately by Proposition 1.1.

Proof of Theorem 1. (a) We only need to show $\Gamma_i \supset Z/2 \times Z/2$ for $2 \le i \le 5$ by Lemma 1.2. In fact, write $Z/2 \times Z/2 = \langle x, y | x^2 = y^2 = 1,\ xy = yx\rangle$ in the following forms:

(i) when $g = 2$, $Z/2 \times Z/2 = \langle x, x, x, y, xy\rangle$, $h = 0$, $b = 5$;
(ii) when $g = 3$, $Z/2 \times Z/2 = \langle x; y; x, x\rangle$, $h = 1$, $b = 2$;
(iii) when $g = 4$, $Z/2 \times Z/2 = \langle x; y; x, y, xy\rangle$, $h = 1$, $b = 3$;
(iv) when $g = 5$, $Z/2 \times Z/2 = \langle x; y; x, y, x, y\rangle$, $h = 1$, $b = 4$.

It is straightforward to check by Proposition 1.1 that (i) $\Gamma_2 \supset Z/2 \times Z/2$ with branching data $(0; 2, 2, 2, 2, 2)$, (ii) $\Gamma_3 \supset Z/2 \times Z/2$ with branching data $(1; 2, 2)$, (iii) $\Gamma_4 \supset Z/2 \times Z/2$ with branching data $(1; 2, 2, 2)$, and (iv) $\Gamma_5 \supset Z/2 \times Z/2$ with branching data $(1; 2, 2, 2, 2)$. We have proved that Γ_2 is never 2-periodic.

In the rest of this paper, p is an odd prime.

(b) Γ_{kp+i} is p-periodic for $i \ne 1$. Otherwise, we have $\Gamma_{kp+i} \supset Z/p \times Z/p$. The Riemann-Hurwitz formula holds: $2(kp+i)-2 = p^2(2h-2)+p^2(1-1/p)b$, i.e., $2k+(2i-2)/p = p(2h-2)+(p-1)b$ implies $2i-2 = 0 \mod(p)$, forcing $i = 1 \mod(p)$. This is a contradiction.

(c) *Claim* 1. If $k = 0, -1 \mod(p)$ or the interval $[(2k+3)/p, (2k+2)/(p-1)]$ contains an integer, then $\Gamma_{kp+1} \supset Z/p \otimes Z/p$.

Case 1. $k = 0 \mod(p)$. Suppose $k = np$, where n is a nonnegative integer. We show a $Z/p \times Z/p$ free action on S_{kp+1} by Proposition 1.1. In fact, write $Z/p \times Z/p = \langle x, y | x^p = y^p = 1\ xy = yx \rangle = \langle x_1, x_2, \dots, x_{n+1}; y_1, y_2, \dots, y_{n+1} \rangle$, $h = 0$, $b = n+1$, where $x_i = x$ and $y_i = y$, $1 \le i \le n+1$. Notice that the Riemann-Hurwitz formula $2(kp+1)-2 = p^2(2(n+1)-2)$ holds.

Case 2. $k = -1 \mod(p)$. We show that there is a $Z/p \times Z/p$ action with two singular points on S_{kp+1} by Proposition 1.1. Write

$$Z/p \times Z/p = \langle x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n; x, x^{-1} \rangle,$$

$h = n$, $b = 2$, where $n = (k+1)/p \ge 1$, $x_i = x$, $y_i = y$, $1 \le i \le n$. Notice that the Riemann-Hurwitz formula $2(kp+1) - 2 = p^2 2(n-1) + p^2 2(1-1/p)$ holds.

Case 3. The interval $[(2k+3)/p, (2k+2)/(p-1)]$ contains an integer n. We show that there is a $Z/p \times Z/p$ action on S_{kp+1} with $t = np - 2k$ singular points. Note $t = np - 2k \ge 3$.

Let $h = k+1-n(p-1)/2$, then $h \ge k+1-(2k+2)/2 = 0$. Write $Z/p \times Z/p = \langle x_1, x_2, \dots, x_h;\ y_1, y_2, \dots, y_h;\ y, x_1, x_2, \dots, x_{t-2}, (\prod_{1 \le j \le t-2} x_j)^{-1} y^{-1} \rangle$. Here $x_i = x_j = x$ and $y_i = y$, $1 \le i \le h$, $1 \le j \le t-2$. Note that the Riemann-Hurwitz formula $2(kp+1) - 2 = p^2(2h-2) + p^2(1-1/p)t$ holds, i.e., $2kp = p^2(2k - n(p-1)) + p(p-1)(np-2k) = 2kp^2 - np^2(p-1) + np^2(p-1) - 2kp(p-1)$.

Claim 2. Conversely, if Γ_{kp+1} is not p-periodic, then $k = 0 \mod(p)$, $k = -1 \mod(p)$, or the interval $[(2k+3)/p, (2k+2)/(p-1)]$ contains an integer.

Let $\Gamma_{kp+1} \supset Z/p \times Z/p$, i.e., there exists a $Z/p \times Z/p$ action on S_{kp+1}.

Case 1. The $Z/p \times Z/p$ acts freely on S_{kp+1}. Then the Riemann-Hurwitz formula $2(kp+1) - 2 = p^2(2h-2)$ implies $k = p(h-1)$, i.e., $k = 0 \mod(p)$.

Case 2. The $Z/p \times Z/p$ acts on S_{kp+1} with two singular points. Then the Riemann-Hurwitz formula $2(kp+1) - 2 = p^2(2h-2) + p^2(1-1/p)2$ implies $k = p(h-1) + p - 1$, i.e., $k = -1 \mod(p)$.

Case 3. The $Z/p \times Z/p$ acts on S_{kp+1} with more than three singular points. Then the Riemann-Hurwitz formula $2(kp+1) - 2 = p^2(2h-2) + p^2(1-1/p)t$ implies $(2k+t)p = (2h-2+t)p^2$. Suppose $n = 2h-2+t$ and $np = (2h-2+t)p = 2k+t \ge 2k+3$ because of $t \ge 3$, i.e., $n \ge (2k+3)/p$. Also, $2kp - (2h-2)p^2 = p(p-1)t$, or $t = (2k - 2hp + 2p)/(p-1)$ implies

$$\begin{aligned} n = 2h-2+t &= [(2h-2)(p-1) + 2k - 2hp + 2p]/(p-1) \\ &= (2k+2-2h)/(p-1) \le (2k+2)/(p-1) \end{aligned}$$

since $h \ge 0$. So $(2k+3)/p \le n \le (2k+2)/(p-1)$ where n is an integer.

Claim 3. Γ_{kp+1} is not p-periodic if $k \ge (p^2-3)/2$. Since $p \ge 3$, it follows that

$$\begin{aligned} (2k+2)/(p-1) - (2k+3)/p &= (2kp + 2p - 2kp + 2k - 3p + 3)/[p(p-1)] \\ &\ge (p^2 - p)/[p(p-1)] = 1 \end{aligned}$$

implies that there exists at least an integer $n \in [(2k+3)/p, (2k+2)/(p-1)]$.

We have proved Theorem 1.

2. The metacyclic subgroups of the mapping class group

Let $M_{p,k} = \langle a, b | a^{k(p-1)} = 1, b^p = 1, aba^{-1} = b^s \rangle$ and $N_{p,d} = \langle a, b | a^d = 1, b^p = 1, aba^{-1} = b^t \rangle$ denote metacyclic groups, where p is an odd prime, k is a positive integer, $d \geq 3$ and divides $p-1$, s and t are $\mathrm{mod}(p)$ integers such that the order of s in the multiple group $(Z/pZ)^*$ is equal to $p-1$, and the order of t in the multiple group $(Z/p)^*$ is equal to d. Then the order $|M_{p,k}| = kp(p-1)$ and $|N_{p,d}| = pd$.

Lemma 2.1. (a) $\Gamma_{(p-1)(kp-2-k)/2} \supset M_{p,k}$ *except for* $k = 1$, $p = 3$.

(b) $\Gamma_{(p-1)(d-2)/2} \supset N_{p,d}$.

Proof. We apply Lemma 1.1 to show $\mathrm{Homeo}^+(S_{(p-1)(kp-2-k)/2}) \supset M_{p,k}$ and $\mathrm{Homeo}^+(S_{(p-1)(d-2)/2}) \supset N_{p,d}$. Write the groups $M_{p,k} = \langle b^{-1}, a, a^{-1}b \rangle$ $(h = 0,\ b = 3)$ and $N_{p,d} = \langle b^{-1}, a, a^{-1}b \rangle$ $(h = 0,\ b = 3)$. Note the order $O(a^{-1}b) = k(p-1)$ in $M_{p,k}$ and the order $O(a^{-1}b) = d$ in $N_{p,d}$. It is direct to check that the Riemann-Hurwitz formula holds for both groups $M_{p,k}$ and $N_{p,d}$:

$$\begin{aligned} 2(p-1)(kp-2-k)/2 - 2 &= kp(p-1)(2(0)-2) + kp(p-1)(1-1/p)1 \\ &\quad + kp(p-1)(1-1/[k(p-1)])/2, \\ 2(p-1)(d-1)/2 - 2 &= pd(2(0)-2) + pd(1-1/p) + pd(1-1/d)2. \end{aligned}$$

Lemma 2.2. *The finite group* $M_{p,k}$ *(resp.* $N_{p,d}$*) is p-periodic with the p-period* $2(p-1)$ *(resp.* $2d$*) for* $k \neq 0 \mod(p)$.

Proof. The order $|M_{p,k}| = kp(p-1)$ (resp. $|N_{p,d}| = pd$) implies that the group $M_{p,k}$ (resp. $N_{p,d}$) does not contain $Z/p \times Z/p$ except $k = 0 \mod(p)$, i.e., $M_{p,k}$ (resp. $N_{p,d}$) is p-periodic. We will compute the p-period of $M_{p,k}$ (resp. $N_{p,d}$) by using a result of Swan [S] that states that the p-period of a p-periodic finite group equals $2|N(S_p)/C(S_p)|$ for an odd prime p, where $N(-)$ and $C(-)$ denote the normalizer and centralizer and S_p is a p-Sylow subgroup of G. In fact, the order $|N(S_p)/C(S_p)| = |N(Z/p)/C(Z/p)| = kp(p-1)/kp = p-1$ in $M_{p,k}$ and the order $|N(Z/p)/C(Z/p)| = pd/p = d$ in $N_{p,d}$. This completes the proof of Lemma 2.2.

The p-periodicity of $\Gamma_{(p-1)(kp-2-k)/2}$ and $\Gamma_{(p-1)(d-2)/2}$ are clear by Theorem 1(b). So a lower bound of the p-period of certain mapping class group is obtained by combining Lemmas 2.1 and 2.2.

We have thus proved Theorem 3 and the following lemma.

Lemma 2.3. *The p-period of* $\Gamma_{(p-1)(kp-2-k)/2}$ *is a multiple of* $2(p-1)$.

3. The Chern classes of the canonical homology representation of the mapping class group

Recall that for a complex representation $f\colon \Gamma \to \mathrm{GL}(k, C)$ of the discrete group Γ the Chern classes $c_i(f) \in H^{2i}(\Gamma; Z)$ are defined as Chern classes of the flat C^k-bundle over $K(\Gamma, 1)$ classified by $Bf\colon K(\Gamma, 1) \to \mathrm{BGL}(k, C)$. Let Q be a subring in C. If $f\colon \Gamma \to \mathrm{GL}(k, Q)$ is a representation over Q, we will write $c_i(f)$ for the ith Chern class of the associated complex representation $\Gamma \to \mathrm{GL}(k, Q) \to \mathrm{GL}(k, C)$.

It is well known that over Q the group Z/nZ has a unique faithful irreducible representation $\sigma_n\colon Z/nZ \to \mathrm{GL}(\varphi(n), Q)$, where $\varphi(n)$ is the Euler function. Glover and Mislin showed a result in [GM].

Proposition 3.1 (Glover and Mislin). *Let $r\colon Z/p^\alpha \to \mathrm{GL}(k, Q)$ be a Q representation. Suppose that in the decomposition of r into Q irreducible representation $\sigma_p\alpha$ occurs with multiplicity m, where m is not divisible by p. Then for every $j > 0$, $(C_{\varphi(p^\alpha)}(r))^j \in H^{2j\varphi(p^\alpha)}(Z/p^\alpha; Z)$ has order p^α.*

Let $\mu\colon \Gamma_g \to \mathrm{Sp}(2g, Z)$ be the map obtained by allowing a homeomorphism h of S_g to act on $H_1(S_g; Z)$ and let $i\colon \mathrm{Sp}(2g, Z) \to \mathrm{GL}(2g, Q)$ be the canonical inclusion. Then $\varepsilon = i\mu\colon \Gamma_g \to \mathrm{GL}(2g, Q)$ is a representation over Q. If $\rho\colon Z/p \to \Gamma_g \to \mathrm{GL}(2g, Q)$ is the composite of inclusion and ε, and since ρ is faithful, $\chi_\rho = m_\rho\chi_{\mathrm{tr}} + n_\rho\chi_\sigma$, where χ stands for the character of the representation, tr denotes the trivial representation, σ is the unique irreducible representation of Z/p, and the integers m_ρ and n_ρ depend on ρ [Se].

Proposition 3.2. *Suppose Γ_g is p-periodic for an odd prime p, $\rho = \varepsilon i\colon Z/p \to \Gamma_g \to \mathrm{GL}(2g, Q)$ is a representation of Z/p over Q for any inclusion $i : Z/p \to \Gamma_g$, and $\chi_\rho = m_\rho\chi_{\mathrm{tr}} + n_\rho\chi_\sigma$. If n_ρ is not divisible by p where i ranges over all inclusions, then Γ_g has p-period m_p, dividing $2\varphi(p) = 2(p-1)$.*

Proof. It is well known that Γ_g is p-periodic ($p > 2$) if and only if every p-Sylow subgroup S_p of Γ_g is cyclic. We need to show that there exists an element $a \in \widehat{H}^{2\varphi(p)}(\Gamma_g; Z)$ such that $\mathrm{Res}(a) \in \widehat{H}^{2\varphi(p)}(Z/p; Z)$ is nontrivial for every Z/p inclusion by the Brown-Venkov theorem.

Let $g^*\colon H^{2\varphi(p)}(\Gamma_g; Z) \to \widehat{H}^{2\varphi(p)}(\Gamma_g; Z)$ be the canonical map from the ordinary cohomology to the Farrell-Tate cohomology. The following diagram is commutative:

$$\begin{array}{ccc}
\widehat{H}^{2\varphi(p)}(\Gamma_g; Z) & \xrightarrow{\mathrm{Res}} & \widehat{H}^{2\varphi(p)}(Z/p; Z) \\
\uparrow g^* & & \uparrow g^* \\
H^{2\varphi(p)}(\Gamma_g; Z) & \xrightarrow[\mathrm{Res}]{} & H^{2\varphi(p)}(Z/p; Z)
\end{array}$$

Let $a = g^*[c_{p-1}(\varepsilon)] \in \widehat{H}^{2\varphi(p)}(\Gamma_g; Z)$. Then $\mathrm{Res}(a) = \mathrm{Res}\, g^*[c_{p-1}(\varepsilon)] = g^*\,\mathrm{Res}[c_{p-1}(\varepsilon)] = g^*[c_{p-1}(\rho)]$ has nontrivial order in $\widehat{H}^{2\varphi(p)}(Z/p; Z) = H^{2\varphi(p)}(Z/p; Z)$ for every Z/p inclusion since the irreducible representation σ occurs with multiplicity n that is not divisible by p, i.e., the cup-product with $g^*[c_{p-1}(\varepsilon)]$ gives an isomorphism for all integers i and a Γ_g-module Z: $\widehat{H}^i(\Gamma_g; Z)_{(p)} \to \widehat{H}^{2\varphi(p)+i}(\Gamma_g; Z)_{(p)}$.

Next, we give an upper bound of the p-period of $\Gamma_{(p-1)(kp-2-k)/2}$ when $k < (p-1)/2$. Consider any inclusion $i\colon Z/p \to \Gamma_{(p-1)(kp-2-k)/2}$. The Riemann-Hurwitz formula $2(p-1)(kp-2-k)/2 - 2 = p(2h-2) + (p-1)t$ implies $t = kp-k-2ph/(p-1) = k(p-1)-sp$, where $s = 2h/(p-1)$ must be an integer. The Lefschetz-Hopf trace formula also tells us $\chi_\rho(T) = 2-t = 2-k(p-1)+sp$, where T is a generator of Z/p and $\rho = \varepsilon i\colon Z/p \to \mathrm{GL}((p-1)(kp-2-k), Q)$.

Since $\chi_{\mathrm{tr}}(1) = 1$, $\chi_{\mathrm{tr}}(T) = 1$, $\chi_\sigma(1) = p-1$, and $\chi_\sigma(T) = -1$, we have $\chi_\rho(1) = m+n(p-1) = (p-1)(kp-k-2)$ and $\chi_\rho(T) = m-n = 2-k(p-1)+sp$.

Then $np = (p-1)(kp-k-2) - 2 + k(p-1) - sp = (kp-k-2-s)p$ implies $n = kp-k-2-s$. Notice $t \geq 0$ implies $0 \leq s \leq k-1$. If $k < (p-1)/2$, then $(k-1)p < kp-k-k+1-2 \leq kp-k-s-2 = n < kp$. So n is not divisible by p. By Proposition 3.2 we conclude that the p-period of $\Gamma_{(p-1)(kp-2-k)/2}$ divides $2(p-1)$ if $0 < k < (p-1)/2$.

We have now proved Theorem 2 by combining the results of these two sections.

4. An upper bound of the p-period of a p-periodic mapping class group Γ_g

In this section, we construct a homogeneous Chern class polynomial of the canonical homology representation $\Gamma_g \to \mathrm{GL}(2g, Q)$, which depends only upon the genus g and the prime p so that the restrictions of the homogeneous Chern class polynomial to all Z/p inclusions in Γ_g are nontrivial. Therefore an upper bound of the p-period of a p-periodic Γ_g is obtained.

We still suppose $g > 1$ and p an odd prime. Let $2g-2 = mp-i$, where $0 \leq i \leq p-1$. Define a fixed point number set $B_{g,p}$ as follows:

$$B_{g,p} = \{i, i+p, i+2p, \dots, i+([2g/(p-1)]-m)p\} \quad \text{if } i \neq 1,$$
$$B_{g,p} = \{1+p, 1+2p, \dots, 1+([2g/(p-1)]-m)p\} \quad \text{if } i = 1.$$

Remark. Define $B_{g,p} = \varnothing$ if in the cases (a) $i \neq 1$ and $2p/(p-1) < m$ and (b) $i = 1$ and $2g/(p-1) < m+1$.

Lemma 4.1. *If $Z/p = \langle x \rangle$ acts on the surface S_g and $2g-2 = mp-i$, where $0 \leq i \leq p-1$, then the number t of fixed points of x belongs to $B_{g,p}$. Conversely, any number $t \in B_{g,p}$ can occur as the number of fixed points of an order p homeomorphism x on the surface S_g.*

Proof. We again use Proposition 1.1. If $Z/p = \langle x \rangle$ acts on the surface, the Riemann-Hurwitz formula $2g-2 = (2h-2)p + tp(1-1/p)$ implies $t = 2(g-h)/(p-1) - 2h - 2 = n - (2g - n(p-1)) + 2 = np - 2g + 2$. Here $g - h = n(p-1)/2$, n is an integer, and $n \leq [2g/(p-1)]$ since $h \geq 0$. Therefore, $t = -2g+2 = i \mod(p)$ and $0 \leq t \leq ([2g/(p-1)]-m)p + i$, i.e., $t \in B_{g,p}$. Notice, if $i = 1$ then $t \neq 1$, since the number of fixed points of Z/p action cannot be 1.

Conversely, if $t \in B_{g,p}$, i.e., $t = i + kp$, where $0 \leq k \leq [2g/(p-1)] - m$ if $i \neq 1$; $0 < k \leq [2g/(p-1)] - m$ if $i = 1$. Let $n = k+m \leq [2g/(p-1)]$. Then $h = g - n(p-1) = g - (k+m)(p-1)/2 \geq 0$. We show that there exists a $Z/p = \langle x \rangle$ action on the surface S_g with t fixed points numbered by Proposition 1.1. In fact, write

$$Z/p = \langle x \rangle = \langle x_1 \cdots x_h ; x_1^{-1} \cdots x_h^{-1} ; x_{h+1}, \dots, x_{h+t-1}, x^{-(t-1)} \rangle$$

for $t \neq 1 \mod(p)$;

$$Z/p = \langle x \rangle = \langle x_1 \cdots x_h ; x_1^{-1} \cdots x_h^{-1} ; x_{h+1}^2, \dots, x_{h+t-1}, x^{-t} \rangle$$

for $t = 1 \mod(p)$, where $x_j = x$, $0 < j < h+t$; it is easy to check the Riemann-Hurwitz formula $(2h-2)p + (p-1)t = (2g - n(p-1) - 2)p + (p-1)(i+kp) = mp - i = 2g-2$ since $n = k+m$.

Lemma 4.2. *Let $Z/p = \langle x \rangle$ act on the surface S_g and $\rho = \varepsilon i \colon Z_p \to \Gamma_g \to \mathrm{GL}(2g, Q)$ be a representation for an inclusion $i \colon Z_p \to \Gamma_g$. Then ρ is equivalent to one of the following representations as a complex representation: $\rho_k = (m+k)\sigma_p \oplus nTr$, where $2g-2 = mp-i$, $0 \le i \le p-1$. If $i \ne 1$, $0 \le k \le [2g/(p-1)]-m$, if $i = 1$, $1 \le k \le [2g/(p-1)]-m$, $n = 2g-(m+k)(p-1)$. Conversely, any representation of $\rho_k = (m+k)\sigma_p \oplus nTr$ as above can be equivalent to $\rho = \varepsilon i$ for some inclusion $i \colon Z/p \to \Gamma_g$.*

Proof. It is clear that the character $\chi_{\rho_k}(\mathrm{Id}) = (m+k)(p-1)+n = 2g$ and $\chi_{\rho_k}(x) = (m+k)+n = 2g-(m+k)p = 2-i-kp$. On the other hand, by using the Lefschetz fixed point theorem, for any $\rho = \varepsilon i$, $\chi_\rho(\mathrm{Id}) = 2g$, $\chi_\rho(x) = 2-t \in 2-B_{g,p} = \{2-i-kp\}$. Lemma 4.1 implies $\chi_\rho = \chi_{\rho_k}$ for some k. Conversely, by Lemma 4.1, there exists a Z/p representation ρ such that $\chi_\rho = \chi_{\rho_k}$ for every k. This completes the proof of Lemma 4.2.

Lemma 4.3. *Let $2g-2 = mp-i$, where p is an odd prime and $0 \le i \le p-1$. If $m < p^r$ then $[2g/(p-1)] < p^{r+1}-p^r$.*

Proof. We have $2g-2+i = mp < p^{r+1}$, i.e., $2g/(p-1) < (p^{r+1}+2)/(p-1)$. Since $3p^{r+1}+2 \le p^{r+2}+p^r$ implies $p^{r+1}+2 \le p^{r+2}-2p^{r+1}+p^r$, we obtain $(p^{r+1}+2)/(p-1) \le p^{r+1}-p^r$.

Lemma 4.4. *If $1 \le k \le p-1$ and $kp^r \le n < (k+1)p^r$, then $n!/p^r!(n-p^r)! = k \bmod(p)$.*

Proof. Write the integer $n!/p^r!(n-p^r)! = [(p^r+1)/(1)][(p^r+2)/(2)] \times [(p^r+3)/(3)] \cdots [(p^r+p)/(p)][(p^r+p+1)/(p+1)] \cdots [(p^r+p^2)/(p^2)] \cdots \times [(p^r+p^r)/(p^r)][(p^r+p^r+1)/(p^r+1)] \cdots [(p^r+2p^r)/(2p^r)][(p^r+2p^r+1)/(2p^r+1)] \cdots [(p^r+3p^r)/(3p^r)] \cdots [(kp^r)/(k-1)p^r] \cdots [(n)/(n-p^r)]$.

If $i \ne 0 \bmod(p)$ then $[(p^r+i)/(i)] = 1$ in the field F_p.

If $i = 0 \bmod(p^{s-1})$ and $i \ne 0 \bmod(p^s)$ $(s \le r)$, then $[(p^r+i)/(i)] = 1$ in the field F_p.

If $i = mp^r$, $m = 0, 1, 2, \ldots, k-1$, then $[(p^r+i)/(i)] = [(m+1)/(m)]$ in F_p.

So $n!/p^r!(n-p^r)! = [(1)][(1)][(2)/(1)] \cdots [(k)/(k-1)] = [(k)]$ in F_p, i.e., the integer $n!/p^r!(n-p^r)! = k \bmod(p)$ when $kp^r \le n < (k+1)p^r$.

Proof of Theorem 4. (a) If $[2g/(p-1)] < p^r$, let $\rho_k \colon Z/p \to \Gamma_g \to \mathrm{GL}(2g, Q)$ be a representation equivalent to the complex $\rho_k = (m+k)\sigma_p \oplus nTr$. The total Chern class C satisfies

$$\begin{aligned} C(\rho_k) &= C(\sigma_p)^{m+k} = [1 + c_{p-1}(\sigma_p)]^{m+k} \\ &= \sum_{0 \le t \le m+k} (m+k)!/t!(m+k-t)![c_{p-1}(\sigma_p)]^t . \end{aligned}$$

So

$$i^* c_{\varphi(p^r)}(\varepsilon) = c_{\varphi(p^r)}(\rho_k) = (m+k)!/p^{r-1}!(m+k-p^{r-1})![c_{p-1}(\sigma_p)]^{p^{r-1}} \ne 0 \mod(p)$$

in $H^{2p^{r-1}(p-1)}(Z/p\,;Z)$ for every inclusion $i\colon Z/p \to \Gamma_g$ since $p^{r-1} \le m+k \le [2g/(p-1)] < p^r$ by Lemma 4.4. Note that Γ_g is p-periodic by assumption. Using an argument similar to the proof of Proposition 3.2, we obtain an upper bound $2p^{r-1}(p-1)$ of the p-period of Γ_g in this case.

(b) If $[2g/(p-1)] \ge p^r$, consider the restriction of a homogeneous Chern class polynomial to every Z/p inclusion. We have

$$\begin{aligned} &i^*\{[c_{\varphi(p^r)}(\varepsilon)]^{p(p-1)}[c_{\varphi(p^{r+1})}(\varepsilon)]^{p-1}\} \\ &\quad = [c_{\varphi(p^r)}(\rho_k)]^{p(p-1)} + [c_{\varphi(p^{r+1})}(\rho_k)]^{p-1} \\ &\quad = [(m+k)!/p^{r-1}!(m+k-p^{r-1})!]^{p(p-1)}[c_{p-1}(\sigma_p)]^{p^r(p-1)} \\ &\qquad + [(m+k)!p^r!(m+k-p^r)!]^{p-1}[c_{p-1}(\sigma_p)]^{p^r(p-1)} \end{aligned}$$

in $H^{2p^r(p-1)^2}(Z/p\,;Z)$.

If $m+k < p^r$, the second term above vanishes and the first term is nontrivial by Lemma 4.4 since $p^{r-1} \le m+k < p^r$. If $m+k \ge p^r$, then $m+k \le [2g/(p-1)] < p^{r+1}-p^r$ by Lemma 4.3. It implies that the second term above always equals $1 \mod(p)$ and the first term above is 0 or $1 \mod(p)$. So, the element $i^*\{[c_{\varphi(p^r)}(\varepsilon)]^{p(p-1)}+[c_{\varphi(p^{r+1})}(\varepsilon)]^{p-1}\} = [c_{\varphi(p^r)}(\rho_k)]^{p(p-1)}+[c_{\varphi(p^{r+1})}(\rho_k)]^{p-1}$ has order p in $H^{2p^r(p-1)^2}(Z/p\,;Z)$ for every Z/p inclusion. It follows that $2p^r(p-1)^2$ is an upper bound of the p-period of a periodic Γ_g, and we have completed the proof of Theorem 4.

Remark. The upper bound of the p-period of Γ_g in Theorem 4 is a little bit rough. It can be improved by individually computing the Chern classes of the homology representation of Γ_g with the same method.

Example. Consider the 3-periodic group Γ_3, $i\colon Z/3 \to \Gamma_3$ is an inclusion, the number of possible fixed points are 2 or 5; i.e., the associated representations are $\rho_1 = 2\sigma_3 \oplus 2Tr$ or $\rho_2 = 2\sigma_3$. But $i_1 * c_2(\varepsilon) = c_2(\rho_1) = c_2(2\sigma_3 \oplus 2Tr) = 2c_2(\sigma_3)$, $i * c_6(\varepsilon) = c_6(\rho_1) = c_6(2\sigma_3 \oplus 2Tr) = 0$. Therefore, $i_1 * \{[c_2(\varepsilon)]^3 + c_6(\varepsilon)\} = 2[c_2(\sigma_3)]^3$ is nontrivial. Similarly, $i_2 * c_2(\varepsilon) = c_2(\rho_2) = c_2(3\sigma_3) = 3c_2(\sigma_3) = 0 \mod(3)$, $i_2 * c_6(\varepsilon) = c_6(\rho_2) = c_6(3\sigma_3) = [c_2(\sigma_3)]^3$. So, $i_2 * \{[c_2(\varepsilon)]^3 + c_6(\varepsilon)\} = c_2(\sigma_3)$ is nontrivial, i.e., the element $[c_2(\varepsilon)]^3+c_6(\varepsilon)$ is nontrivial when restricted to every $Z/3$ subgroup. Thus, we obtain an upper bound 12 of the 3-period of Γ_3, which is better than the upper bound 24 given by Theorem 4.

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