COMMUTATOR IDEALS AND SEMICOMMUTATOR IDEALS OF TOEPLITZ ALGEBRAS ASSOCIATED WITH FLOWS

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ABSTRACT. We prove that for a Toeplitz C^*-algebra associated with a flow that has no fixed points, the commutator ideal and the semicommutator ideal coincide.

INTRODUCTION

This is a continuation of our study of Toeplitz algebras associated with flows [2, 6, 8, 9]. Consider the C^*-algebra $\mathscr{T}(C)$ generated by Toeplitz operators $\{T_\varphi : \varphi \in C\}$ on $H^2(\mathbb{R})$, where C is a C^*-algebra of bounded, uniformly continuous functions on $\mathbb{R}$ that is closed under the natural translation. This algebra has been studied in various contexts. Two of the most important aspects concerning $\mathscr{T}(C)$ are its ideal structure and generalized Fredholm theory for operators in $\mathscr{T}(C)$. In either case, one inevitably encounters the ideal $\mathscr{C}_{1/2}(C)$ of $\mathscr{T}(C)$ generated by semicommutators $\{T_\varphi T_\psi - T_{\varphi\psi} : \varphi, \psi \in C\}$. As it turns out, the quotient algebra $\mathscr{T}(C)/\mathscr{C}_{1/2}(C)$ is always isomorphic to C itself. After the identification of this quotient algebra, one can proceed to establish an index theory that identifies topological invariants of C with analytic invariants of a von Neumann algebra into which $\mathscr{T}(C)$ is embedded [2, 3]. All this is very nice until one develops the uneasy feeling that there might be something unnatural involved here. The source of this uneasiness is the semicommutator ideal $\mathscr{C}_{1/2}(C)$. As it is defined, $\mathscr{C}_{1/2}(C)$ is not a natural algebraic object; its definition depends on properties that only Toeplitz operators possess. One cannot help feeling that $\mathscr{C}_{1/2}(C)$ is too much of an ad-hoc concoction. In contrast, the ideal $\mathscr{C}(C)$ of $\mathscr{T}(C)$ generated by *commutators* seems to be a much more natural algebraic object. After all, every C^*-algebra has a commutator ideal, and the quotient is, of course, a commutative C^*-algebra. Because of the naturalness of $\mathscr{C}(C)$, it is more desirable to establish an index theory for $\mathscr{T}(C)$ corresponding to the exact sequence

$$0 \to \mathscr{C}(C) \to \mathscr{T}(C) \to \mathscr{T}(C)/\mathscr{C}(C) \to 0.$$

However, the trouble with this possible index theory is that, except for the few

cases where $\mathscr{C}(C)$ and $\mathscr{C}_{1/2}(C)$ are known to coincide, the quotient algebra $\mathscr{T}(C)/\mathscr{C}(C)$ is not calculable for an arbitrary flow.

So the situation we face is that, on the one hand, we can establish a nice index theory for $\mathscr{T}(C)$ with an apparently unnatural ideal while, on the other hand, we have a perfectly natural ideal that has little use in the study of the Toeplitz algebra. This calls for a thorough investigation of the relation between the two ideals. The most tempting thought is that it would be nice if these ideals turned out to be the same. But examples tell us that this is not the case. However, it is quite easy to show that if the flow induced by the natural translation on C is uniquely ergodic, then the two ideals are the same [2]. But even at the time when [2] was being written, we had the feeling that the $\mathscr{C}_{1/2}(C)$ and $\mathscr{C}(C)$ should coincide for a much larger class of flows. This class, we felt, should include, at the very least, every minimal flow. The purpose of this note is to show that for any flow without fixed points, these ideals coincide.

1. THE MAIN RESULT

Let X be a compact Hausdorff space upon which $\mathbb{R}$ acts as a group of homeomorphisms. Write $x + t$ for the translation of $x \in X$ by $t \in \mathbb{R}$. Assume that $(x, t) \mapsto x + t$ is a continuous map from $X \times \mathbb{R}$ to X. We refer to the action of $\mathbb{R}$ on X as a *flow*. Let $A(X)$ denote the collection of continuous functions φ on X such that for every $x \in X$, the function $t \mapsto \varphi(x + t)$ belongs to $H^\infty(\mathbb{R})$.

Definition. An *analytic representation* of the flow $(X, \mathbb{R})$ is a pair (π, P) where π is a C^*-algebra representation of $C(X)$ on some Hilbert space K and P is an orthogonal projection on K satisfying

(1.1) $\pi(1) = 1$,
(1.2) $\pi(\varphi)P = P\pi(\varphi)P$ for every $\varphi \in A(X)$.

We say that an analytic representation (π, P) of the flow is *faithful* if the C^*-algebra $\mathscr{A}$ generated by $\pi(C(X))$ and P is noncommutative.

The reason for the term "faithful" is that in [9] it is shown that there exists a universal C^*-algebra with the property that there is a bijective correspondence between C^*-representations of it and analytic representations of the flow; and an analytic representation gives rise to a faithful C^*-representation if and only if it is faithful in the sense just defined.

Let an analytic representation (π, P) of the flow be given. Let $\mathscr{T}$ denote the Toeplitz algebra associated with (π, P). That is, $\mathscr{T} = P\mathscr{A}P$. $\mathscr{T}$ is regarded both as a subalgebra of $\mathscr{A}$ and as a unital C^*-algebra on the subspace PK. Let $\mathscr{C}$ denote the ideal in $\mathscr{T}$ generated by the commutators $\{AB - BA : A, B \in \mathscr{T}\}$. $\mathscr{C}$ is the same as the ideal of $\mathscr{T}$ generated by

$$\{P\pi(\varphi)P\pi(\psi)P - P\pi(\psi)P\pi(\varphi)P : \varphi, \psi \in C(X)\}.$$

Let $\mathscr{C}_{1/2}$ denote the ideal of $\mathscr{T}$ generated by the semicommutators

$$\{P\pi(\varphi)P\pi(\psi)P - P\pi(\varphi\psi)P : \varphi, \psi \in C(X)\}.$$

Obviously $\mathscr{C}_{1/2} \supset \mathscr{C}$. Despite having similar appearances, $\mathscr{C}_{1/2}$ and $\mathscr{C}$ do not always coincide. We first present an example where they differ. This example derives from a well-known result of Gohberg and Krupnik [5], but in their paper,

the setting is quite different. They do not regard their Toeplitz algebras as one associated with a flow but, rather, as one associated with the unit disc.

Example. Consider $C_{*,*}(\mathbb{R})$, the collection of continuous functions f on $\mathbb{R}$ such that the limits $\lim_{s \to +\infty} f(s)$ and $\lim_{s \to -\infty} f(s)$ exist but are not necessarily equal. Let X be the maximal ideal space of $C_{*,*}(\mathbb{R})$. It is elementary that X is the two-point compactification $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ of $\mathbb{R}$. The natural translation of functions induces a flow $\{\alpha_t : t \in \mathbb{R}\}$ on X such that for x in the subset $\mathbb{R}$ of $X = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, $\alpha_t(x) = x + t$ and α_t fixes both $-\infty$ and $+\infty$. Let $K = L^2(\mathbb{R})$ and let P be the projection from $L^2(\mathbb{R})$ onto $H^2(\mathbb{R})$. Further, let $\pi \colon C(X) \to \mathscr{L}(L^2(\mathbb{R}))$ be the representation

$$\pi(\hat{f})g = P(fg), \qquad g \in L^2(\mathbb{R}),$$

for every $f \in C_{*,*}(\mathbb{R})$. ($\hat{f}$ is the Gelfand transform of f.) This pair (π, P) is a nondegenerate analytic representation of the flow. By the results of [5], for this pair of (π, P), we have $\mathscr{C}_{1/2}/\mathscr{C} \cong C_0(0, 1)$. As we will see from our main result, the fact that this flow has fixed points is the key here.

It was shown in [2, Lemma 24.1] that if the flow $(X, \mathbb{R})$ is uniquely ergodic, meaning that the flow possesses only one invariant probability measure, then $\mathscr{C}_{1/2} = \mathscr{C}$. The main purpose of this paper is to show that the equality $\mathscr{C}_{1/2} = \mathscr{C}$ holds for a much larger class of flows.

Theorem 1. *If the flow $(X, \mathbb{R})$ has no fixed points then $\mathscr{C}_{1/2} = \mathscr{C}$.*

Let $C_{\mathrm{bu}}(\mathbb{R})$ denote the collection of bounded, uniformly continuous functions on $\mathbb{R}$. For any subset $F \subset C_{\mathrm{bu}}(\mathbb{R})$, let C_F denote the unital C^*-algebra generated by F together with all its translations on $\mathbb{R}$. Let X_F be the maximal ideal space of C_F. Let $(X_F, \mathbb{R})$ be the flow on X_F induced by the natural translation of functions in C_F. This flow is sometimes called the Beboutov flow associated with the family F. It is quite easy to derive a necessary condition for the existence of fixed points for this flow. The following lemma is completely elementary and so we will not present its proof.

Lemma 1.1. *Suppose that the flow $(X_F, \mathbb{R})$ has at least one fixed point. Then there exist sequences $\{t_n\} \subset \mathbb{R}$ and $\{T_n\} \subset (0, \infty)$ such that*

$\lim_{n \to \infty} |t_n| = \infty$,

$\lim_{n \to \infty} T_n = \infty$, *and*

$\lim_{n \to \infty} \sup\{|g(s) - g(t)| : s, t \in [t_n - T_n, t_n + T_n]\} = 0$

for every $g \in F$.

Consequently, if F contains a function h with the property that there exist $c > 0$ and $r > 0$ such that for any $t \in \mathbb{R}$, $h([t-c, t+c])$ is not contained in any disc of radius r, then the flow $(X_F, \mathbb{R})$ has no fixed points. There are many elementary functions that are uniformly continuous and have this property. For example, with $a \neq 0$,

$$h(t) = \sin(at + b)$$

has this property as do each of the following functions:

$$h(t) = \sin(at + b\log(1 + |t|) + c),$$
$$h(t) = \sin(at + b_1\log(1 + |t|) + b_2|t|^{1/2} + b_3\log(\log(1 + |t|))),$$
$$h(t) = \cos(at + b\sin(\log(1 + |t|^{4/5}))).$$

If F contains *any* of these functions, then by Theorem 1, the commutator ideal and the semicommutator ideal of the usual Toeplitz operators $\{T_\varphi : \varphi \in C_F\}$ on $H^2(\mathbb{R})$ are the same.

The example we gave earlier shows that if the flow has at least *two* fixed points, then $\mathscr{C}_{1/2}$ and $\mathscr{C}$ may differ. At this point it is natural to ask what happens when the flow has only one fixed point? Consider the natural flow on the *one* point compactification of $\mathbb{R}$, i.e., the flow that is the natural translation on $\mathbb{R}$ and fixes the infinity. For this flow, $\mathscr{C}_{1/2}$ and $\mathscr{C}$ are the same. But does there exist a flow with a single fixed point for which $\mathscr{C}_{1/2}$ and $\mathscr{C}$ differ? Or is it true that $\mathscr{C}_{1/2}$ and $\mathscr{C}$ always coincide if the flow has no more than one fixed point? Another question is: Must $\mathscr{C}_{1/2}$ and $\mathscr{C}$ differ if the flow has two or more fixed points and the space is connected? We do not know the answers to these questions.

The rest of the paper is devoted to the proof of Theorem 1. Obviously both $\mathscr{C}_{1/2}$ and $\mathscr{C}$ may be reduced to $\{0\}$ if the representation (π, P) fails to be faithful. So for the rest of the paper, (π, P) is assumed to be faithful. The proof requires some technical preparation, which we will take care of first.

2. Spectrum and fixed points

For each function $\varphi \in C(X)$, let $\text{sp}(\varphi)$ denote the spectrum, in the sense of spectral synthesis, of the function with respect to the flow $(X, \mathbb{R})$. (Readers not familiar with spectral synthesis are referred to [1, 4, 7, 10] for detailed discussions of the subject.) The following facts are well known:

(2.1) $A(X) = \{\varphi : \varphi \in C(X), \text{sp}(\varphi) \subset [0, \infty)\}$;

(2.2) $\text{sp}(\overline{\varphi}) = -\text{sp}(\varphi)$;

(2.3) $\text{sp}(\varphi\psi) \subset [a+b, \infty)$ if $\text{sp}(\varphi) \subset [a, \infty)$ and $\text{sp}(\psi) \subset [b, \infty)$;

(2.4) $\{\varphi : \varphi \in C(X), \text{sp}(\varphi)$ is bounded set$\}$ is dense in $C(X)$.

For a set $S \subset \mathbb{R}$, let

$$C(X, S) = \{\varphi : \varphi \in C(X), \text{sp}(\varphi) \subset S\}.$$

Let dm be an invariant probability measure on X. The flow on X induces a group of unitary $\{u_t : t \in \mathbb{R}\}$ on $L^2(X, dm)$; that is, $(u_t g)(x) = g(x + t)$ for every $g \in L^2(X, dm)$. Write $u_t = \exp(it\delta)$, where δ is the infinitesimal generator of the unitary group. The operator δ has the spectral decomposition

$$\delta = \int_{\mathbb{R}} \lambda \, de_\lambda$$

on $L^2(X, dm)$. For any open interval $I \subset \mathbb{R}$, $C(X, S) \subset e(I)L^2(X, dm)$ whenever the closure of S is contained in I. Furthermore,

(2.5) $e(I)L^2(X, dm) = \bigvee \{C(X, S) : \overline{S} \subset I\}$.

Here, the symbol $\bigvee$ means "the closure of the linear span of".

Proposition 2.1. *The following are equivalent*:

   (a) *The flow* $(X, \mathbb{R})$ *has no fixed points.*

   (b) *For any invariant probability measure* dm *of the flow* $(X, \mathbb{R})$, *the spectrum of the infinitesimal generator* δ *of* $\{u_t : t \in \mathbb{R}\}$ *on* $L^2(X, dm)$ *contains* $c\mathbb{Z}$ *with some* $c > 0$.

Proof. (a) $\Rightarrow$ (b) By the spectral analysis of the flow, if $c \in \sigma(\delta)$ then $\sigma(\delta) \supset c\mathbb{Z}$. Therefore it suffices to rule out the possibility that $\sigma(\delta) = \{0\}$. If it were the case that $\sigma(\delta) = \{0\}$, then every u_t is the identity operator. Therefore, for every $\varphi \in C(X)$, $\varphi = u_t\varphi = \varphi$ as elements in $L^2(X, dm)$. Since φ is continuous, $\varphi(x + t) = \varphi(x)$ for every x in the (closed) support of dm; and since φ is chosen arbitrarily from $C(X)$, we conclude that $x = x + t$ for all such x, contradicting the assumption that the flow has no fixed points.

(b) $\Rightarrow$ (a) If the flow had a fixed point x then the point mass at x would be invariant measure for the flow. The L^2 space of such a measure is $\mathbb{C}$. This contradicts (b). $\square$

3. ANALYTIC REPRESENTATIONS OF THE FLOW

Let (π, P) be a nondegenerate analytic representation of the flow on a Hilbert space K. Define

$$K_+ = \bigvee\{\pi(\varphi)PK : \varphi \in C(X)\}$$

and

$$K_- = \bigcap_{\lambda > 0}[\pi(C(X, [\lambda, \infty)))PK]^- .$$

Obviously K_+ is invariant for $\mathscr{A}$. Because of (1.2), (2.1), (2.3), and (2.4), K_- is also invariant for $\mathscr{A}$. Therefore, so is $K' = K_+ \ominus K_-$. Let $\mathscr{C}\mathscr{A}$ be the commutator ideal of $\mathscr{A}$. Then

(3.1) $$\mathscr{C}\mathscr{A} \,|(K \ominus K_+) = \{0\} = \mathscr{C}\mathscr{A}\,|K_- .$$

Indeed, because both K_+ and $K \ominus K_+$ are invariant subspaces, for any $\varphi \in C(X)$, $[\pi(\varphi), P](K \ominus K_+) \subset K_+ \cap (K \ominus K_+)$. Hence $\mathscr{C}\mathscr{A}\,|(K \ominus K_+) = \{0\}$. On the other hand, it follows from (1.2) and (2.3) that $[\pi(\varphi), P]\,|\,[\pi(\psi)PK]^- = 0$ if $\mathrm{sp}(\varphi) \subset [a, \infty)$ and $\mathrm{sp}(\psi) \subset (\max\{-a, 0\}, \infty)$. Therefore, $\mathscr{C}\mathscr{A} \mid K_- = \{0\}$. K_- and $K \ominus K_+$ should be thought of as the "distant future" and "distant past" for the representation (π, P).

Let π' be the restriction of π of K'; i.e., let $\pi'(\varphi) = \pi(\varphi) \mid K'$ for every $\varphi \in C(X)$. Define $P' = (P - P_-) \mid K'$, where P_- is the orthogonal projection onto K_-. Our analysis shows that (π', P') is a nondegenerate analytic representation of the flow. Furthermore, the pair (π', P') has the properties that

$$\bigvee\{\pi'(\varphi)P'K' : \varphi \in C(X)\} = K'$$

and

$$\bigcap_{\lambda > 0}[\pi'(C(X, [\lambda, \infty)))P'K']^- = \{0\} .$$

In conformity with [9], which in turn was motivated in part by considerations from dilation theory, we call the restriction of (π, P) to K' *pure*. Because of (3.1), $\mathscr{A} \mid (K \ominus K')$ is a commutative C^*-algebra. Thus every analytic representation of a flow decomposes as the direct sum of a pure representation and one whose image is abelian. Moreover, the pure summand is nonzero if and only if the representation is faithful. For our purpose here, only the pure part of (π, P) plays a role.

Suppose that the analytic representation (π, P) is pure so that the equations

(3.2) $$\bigvee\{\pi(\varphi)PK\colon \varphi \in C(X)\} = K$$

and

(3.3) $$\bigcap_{\lambda>0}[\pi(C(X, [\lambda, \infty)))PK]^- = \{0\}$$

are satisfied. For each $\lambda \in \mathbb{R}$, let P_λ denote the orthogonal projection from K onto $[\pi(C(X, [\lambda, \infty)))PK]^-$. Obviously $P_{\lambda_1} \le P_{\lambda_2}$ if $\lambda_1 > \lambda_2$. It follows from (3.2) and (3.3) that $\lim_{\lambda\to-\infty} P_\lambda = 1$ and $\lim_{\lambda\to+\infty} P_\lambda = 0$. Thus

$$E_\lambda = E((-\infty, \lambda]) = 1 - \sup\{P_s\colon s > \lambda\}, \qquad \lambda \in \mathbb{R},$$

is a resolution of the identity on K. Since $1 \in C(X, [0, \infty)) = A(X)$ and $\pi(A(X))P = P\pi(A(X))P$, we have $P = P_0$. Therefore, $P = P_0 \ge \sup\{P_s\colon s > 0\} = 1 - E((-\infty, 0]) = E((0, \infty))$. On the other hand, for any $\lambda < 0$, $E((\lambda, \infty)) = \sup\{P_s\colon s > \lambda\} \ge P_0 = P$. Since $\inf\{E((\lambda, \infty))\colon \lambda < 0\} = E([0, \infty))$, we have

(3.4) $$E((0, \infty)) \le P \le E([0, \infty)).$$

It follows from (2.3) that $\pi(C(X, [\mu, \infty)))P_\lambda K \subset P_{\lambda+\mu}K$ for any $\lambda, \mu \in \mathbb{R}$. Therefore, for any $\lambda, \mu \in \mathbb{R}$ and $\varphi \in C(X, [\mu, \infty))$,

(3.5) $$\pi(\varphi)E((\lambda, \infty))K = \pi(\varphi)\left[\bigvee\{P_\gamma K\colon \gamma > \lambda\}\right] \subset \bigvee\{P_{\gamma+\mu}\colon \gamma > \lambda\}$$
$$= E((\lambda+\mu, \infty))K$$

and

(3.6) $$\pi(\varphi)E([\lambda, \infty))K = \pi(\varphi)\left[\bigcap_{\gamma<\lambda} E((\gamma, \infty))K\right] \subset \bigcap_{\gamma<\lambda} E((\gamma+\mu, \infty))K$$
$$= E([\lambda+\mu, \infty))K.$$

Let

(3.7) $$D = \int_\mathbb{R} \lambda \, dE_\lambda$$

and let

(3.8) $$V(t) = \exp(itD), \qquad t \in \mathbb{R}.$$

The following lemma is known as Forelli's Spectral-Commutation Theorem. It is a consequence of more general results whose proofs may be found in [1, 4, 7, 10]. A direct, elementary proof is presented in [9].

Lemma 3.1. *For every $\varphi \in C(X)$ and every $t \in \mathbb{R}$, $V(t)\pi(\varphi)V(-t) = \pi(\varphi_t)$, where $\varphi_t(x) = \varphi(x+t)$.*

Let $\lambda_0 \in \sigma(D)$, the spectrum of D, pick a sequence of unit vectors $y_n \in E((\lambda_0 - 1/n, \lambda_0 + 1/n))$, let L be a Banach generalized limit on $l^\infty(\mathbb{Z}_+)$, and define

$$\rho(A) = L(\{\langle Ay_n, y_n\rangle\}_{n=1}^\infty)$$

for every bounded operator A on K. Obviously ρ is a state on $\mathscr{L}(K)$.

Lemma 3.2. *There exists an invariant probability measure dm of the flow such that*

$$\rho(\pi(\varphi)) = \int_X \varphi(x)\,dm(x)$$

for every $\varphi \in C(X)$.

Proof. It suffices to show that the functional $\varphi \mapsto \rho(\pi(\varphi))$ is translation invariant, i.e., $\rho(\pi(\varphi)) = \rho(\pi(\varphi_t))$ for every $t \in \mathbb{R}$. By Lemma 3.1, we have

$$\rho(\pi(\varphi_t)) = \rho(V(t)\pi(\varphi)V(-t)) = L(\{\langle \pi(\varphi)V(-t)y_n,\, V(-t)y_n\rangle\}_{n=1}^\infty).$$

Because $y_n \in E((\lambda_0 - 1/n,\, \lambda_0 + 1/n))K$,

$$\|V(-t)y_n - e^{-i\lambda_0 t}y_n\| \le \max_{|\lambda|\le 1/n}|e^{i\lambda t} - 1| \to 0 \quad \text{as } n \to \infty.$$

It follows from the properties of Banach generalized limits that

$$L(\{\langle \pi(\varphi)V(-t)y_n,\, V(-t)y_n\rangle\}_{n=1}^\infty)$$
$$= L(\{\langle \pi(\varphi)e^{-i\lambda_0 t}y_n,\, e^{-i\lambda_0 t}y_n\rangle\}_{n=1}^\infty) = \rho(\pi(\varphi)). \quad \square$$

Lemma 3.3. *If the analytic representation (π, P) of the flow is faithful, then there exists a $\varphi \in A(X)$ such that $P\pi(\varphi)(1-P)\pi(\overline{\varphi})P \ne 0$.*

Proof. We may, of course, assume that (π, P) also satisfies (3.2) and (3.3). Let dm be the measure provided by the previous lemma. Let $\delta = \int_{\mathbb{R}} \lambda\,de_\lambda$ be the spectral decomposition of the infinitesimal generator of the unitary group $\{u_t : t \in \mathbb{R}\}$ on $L^2(X, dm)$. By Proposition 2.1, $\sigma(\delta) \cap (|\lambda_0| + 3, \infty) \ne \varnothing$. Therefore, by (2.5), there exist a $p > |\lambda_0| + 3$ and a $\xi \in A(X)$ such that $\mathrm{sp}(\xi) \subset (p, p+1)$ and $\int_X |\xi|^2\,dm = 1$. By (3.4) and (3.5), $\pi(\xi)y_n \in E((|\lambda_0| + 3 + \lambda_0 - 1, \infty))K \subset E((2, \infty))K \subset PK$. Hence for any $A \in \mathscr{L}(K)$,

(3.9) $$\rho(\pi(\overline{\xi})PAP\pi(\xi)) = \rho(\pi(\overline{\xi})A\pi(\xi)).$$

By (2.1) and (2.3), $\mathrm{sp}(\xi^2) \subset (2p, \infty)$ and $\mathrm{sp}(\overline{\xi}) \subset (-p-1, \infty)$. Therefore, $\mathrm{sp}(\xi^2\overline{\xi}) \subset (p-1, \infty)$ and $\pi(\xi^2\overline{\xi})E((-1, \infty))K \subset E((p-2, \infty))K \subset E((\lambda_0+1, \infty))K$. Consequently $\pi(\overline{\xi}^2\xi)y_n \in E((-\infty, -1])K \subset (1-P)K$. Hence

(3.10) $$(1 - P)\pi(\overline{\xi}^2\xi)y_n = \pi(\overline{\xi}^2\xi)y_n.$$

Combining (3.9), (3.10), and Lemma 3.2, we have

$$\rho(\pi(\overline{\xi})P\pi(\xi^2)(1-P)\pi(\overline{\xi}^2)P\pi(\xi)) = \rho(\pi(\overline{\xi}\xi^2)\pi(\overline{\xi}^2\xi)) = \int_X |\xi|^6\,dm.$$

Because $\int_X |\xi|^2\,dm = 1$, we have $\int_X |\xi|^6\,dm \ne 0$. Hence $P\pi(\xi^2)(1-P)\pi(\overline{\xi}^2)P \ne 0$. $\square$

4. The proof of Theorem 1

For each ideal $\mathscr{I} \subset \mathscr{T}$, let $\widehat{\mathscr{I}}$ denote the ideal in $\mathscr{A}$ generated by $\{ABC : A, C \in \mathscr{A},\ B \in \mathscr{I}\}$. Since $P\mathscr{A}P = \mathscr{T}$, we have $P\widehat{\mathscr{I}}P = \mathscr{I}$. It is obvious that $P\mathscr{C}\mathscr{A}P \subset \mathscr{C}_{1/2}$. On the other hand, because $P\pi(\varphi)P\pi(\psi)P - P\pi(\varphi\psi)P = P([\pi(\varphi)P - P\pi(\varphi)]\pi(\psi))P$ and $\pi(\varphi)P - P\pi(\varphi) \in \mathscr{C}\mathscr{A}$, we have $P\mathscr{C}\mathscr{A}P \supset \mathscr{C}_{1/2}$. Therefore $P\mathscr{C}\mathscr{A}P = \mathscr{C}_{1/2}$. Suppose now that $\mathscr{C}_{1/2} \ne \mathscr{C}$. Then, according to the

above equality, $\widehat{\mathscr{C}}$ would be a proper ideal in $\mathscr{C}\mathscr{A}$ and, consequently, $\mathscr{A}/\widehat{\mathscr{C}}$ would be noncommutative. Let $\Phi\colon \mathscr{A}/\widehat{\mathscr{C}} \to \mathscr{L}(\widehat{K})$ be a faithful representation of $\mathscr{A}/\widehat{\mathscr{C}}$ on a Hilbert space $\widehat{K}$. Because $\mathscr{A}/\widehat{\mathscr{C}}$ is noncommutative, the pair $\hat{\pi}(\varphi) = \Phi(\pi(\varphi) + \widehat{\mathscr{C}})$ and $\widehat{P} = \Phi(P + \widehat{\mathscr{C}})$ gives a nondegenerate analytic representation of the flow. According to Lemma 3.3, there exists a $\varphi \in A(X)$ such that $\widehat{P}\hat{\pi}(\varphi)(1 - \widehat{P})\hat{\pi}(\overline{\varphi})\widehat{P} \neq 0$. On the other hand, since $\varphi \in A(X)$, we have

$$P\pi(\varphi)(1 - P)\pi(\overline{\varphi})P = P\pi(\overline{\varphi})P\pi(\varphi) - P\pi(\overline{\varphi})P\pi(\varphi)P \in \mathscr{C} \subset \widehat{\mathscr{C}}.$$

Therefore

$$\widehat{P}\hat{\pi}(\varphi)(1 - \widehat{P})\hat{\pi}(\overline{\varphi})\widehat{P} = \Phi(P\pi(\varphi)(1 - P)\pi(\overline{\varphi})P + \widehat{\mathscr{C}}) = 0.$$

This contradiction proves the theorem. □

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