RIGIDITY THEOREMS FOR NONPOSITIVE EINSTEIN METRICS

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Abstract. In this paper we study the following problem: When must a complete Einstein metric $g$ on an $n$-manifold with $\text{Ric} = (n - 1)\lambda g$, $\lambda \leq 0$, be a metric of constant curvature $\lambda$?

1. Introduction and main results

Let $g$ be a complete riemannian metric on an $n$-manifold $M$. Denote by $R$ the curvature tensor of $g$. The Ricci curvature $\text{Ric}$ is then defined as

$$\text{Ric}(x, y) = \sum_{i=1}^{n} g(R(x, e_i)e_i, y), \quad x, y \in T_p M,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $T_p M$. The metric $g$ is said to be Einstein if the Ricci curvature is constant, i.e.,

$$\text{Ric} = (n - 1)\lambda g$$

for some constant $\lambda$. $\lambda$ is called the Einstein constant of $g$. It is clear that in dimension three the metric $g$ is Einstein with $\text{Ric} = (n - 1)\lambda g$ if and only if it has constant curvature $\lambda$, i.e.,

$$R(x, y)z = \lambda(g(y, z)x - g(x, z)y), \quad x, y, z \in T_p M.$$ 

In higher dimensions, this is not the case. One may ask if an Einstein metric has constant curvature whenever it has almost constant curvature in a certain sense. From now on we always assume that $g$ is a complete Einstein metric with Einstein constant $\lambda$. It is natural to consider the new tensor $\hat{R}$, defined...
by
\[
\hat{R}(x, y)z = R(x, y)z - \lambda(g(y, z)x - g(x, z)y), \quad x, y, z \in T_pM.
\]

Denote by $\sigma$ the pointwise norm of $\hat{R}$, defined by
\[
\sigma = \sqrt{\sum_{ijkl} g(\hat{R}(e_i, e_j)e_k, e_l)^2},
\]
where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $T_pM$. By a formula in [H], one can easily show that $\sigma$ satisfies
\[
\Delta \sigma + c_0(n)\sigma^2 - 2(n - 1)\lambda \sigma \geq 0
\]
in the sense of distribution, where $c_0(n)$ is a positive constant depending only on $n$ and $\Delta$ denotes the Laplace-Beltrami operator (in $\mathbb{R}^n$, $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$).

In the case of $\lambda > 0$, Myers’s Theorem (cf., e.g., [CE]) tells us that the manifold is closed. Thus by integrating (1), one obtains

**Theorem 1** (Berger [B]). Given $n$, there is a small constant $\varepsilon = \varepsilon(n) > 0$ depending only on $n$ such that if a complete Einstein metric $g$, with $\lambda > 0$, on an $n$-manifold satisfies $\sigma \leq \lambda \varepsilon$, then $\sigma \equiv 0$, i.e., $g$ has constant curvature $\lambda$.

In [S] the author gives an $L^\frac{3}{2}$-version of Theorem 1, which says that if
\[
\int \sigma^{\frac{3}{2}} \leq \lambda \frac{3}{2} \text{vol}(M) \varepsilon
\]
for some small $\varepsilon = \varepsilon(n) > 0$, depending only on $n$, where $\text{vol}(M)$ denotes the volume of $(M, g)$, then $\sigma \equiv 0$.

In the case of $\lambda \leq 0$, the manifold can be compact or noncompact. First let us consider the case $\lambda = 0$. In this case, the following fact is known: There is a small constant $\varepsilon = \varepsilon(n) > 0$ depending only on $n$, if a Ricci-flat metric $g$ on a closed $n$-manifold satisfies
\[
\sigma \cdot \text{dia}(M)^2 \leq \varepsilon,
\]
where $\text{dia}(M)$ denotes the diameter of $g$, then $\sigma \equiv 0$, i.e., $g$ is flat. The proof of this fact is trivial. By a theorem of Gromov [G], any almost flat manifold is aspherical, i.e., its universal cover is diffeomorphic to $\mathbb{R}^n$. Thus for a sufficiently small $\varepsilon = \varepsilon(n)$, (2) implies that the universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$. On the other hand, by the Cheeger-Gromoll’s Splitting Theorem (cf. [CG]), $\tilde{M}$ with the induced metric $\tilde{g}$ is isometric to a riemannian product $N \times \mathbb{R}^k$ for some closed riemannian manifold $N$. Thus $N$ must be a point and $\tilde{g}$ is flat. Therefore $g$ is flat. This argument in fact shows that all nonnegatively Ricci-curved aspherical manifolds are flat. By a theorem of Fukaya-Yamaguchi [FY], if $\text{dia}(M) \leq D$, then condition (2) can be replaced by $-1 \leq K_g \leq \varepsilon$ for a small number $\varepsilon = \varepsilon(n, D) > 0$, where $K_g$ denotes the sectional curvature of $g$.

In §2 we will prove an $L^\frac{3}{2}$-version of the above fact, that is,
Theorem 2. Given $n$, there is a small constant $\varepsilon = \varepsilon(n) > 0$ depending only on $n$ such that if a Ricci-flat metric $g$ on a closed $n$-manifold satisfies

$$\int \sigma^\frac{n}{2} \leq \frac{\text{vol}(M)}{\text{dia}(M)^n} \varepsilon,$$

then $\sigma \equiv 0$, i.e., $g$ is flat.

For complete Ricci-flat metrics on noncompact $n$-manifolds, some rigidity phenomena have been discovered (cf. [A2, Ba, S], etc.). Roughly speaking, if a complete Ricci-flat metric $g$ has sufficiently small total curvature, i.e., there is a small $\varepsilon = \varepsilon(n) > 0$ such that if

$$\int \sigma^\frac{n}{2} \leq \nu^\frac{n+1}{2}_M \varepsilon$$

where

$$\nu_M := \lim_{r \to +\infty} \frac{\text{vol}(B(p, r))}{\sigma^n r^n} > 0,$$

where $\sigma_n$ denotes the volume of the unit ball in $\mathbb{R}^n$, then $\sigma \equiv 0$. It is worth mentioning that the result of Anderson [A2] does not require (3), but requires that $\nu_M \geq 1 - \varepsilon$ for a small $\varepsilon = \varepsilon(n) > 0$.

Now let us consider the case of $\lambda < 0$. The following theorem is first proved by Ye [Y, Theorem 2].

Theorem 3 ([Y]). Given $n$, $D > 0$, and $\lambda < 0$, there is a small constant $\varepsilon = \varepsilon(n, \sqrt{-\lambda}D) > 0$ such that if an Einstein metric $g$, with $\lambda < 0$, on a closed $n$-manifold satisfies $\text{dia}(M) \leq D$ and $\sigma \leq |\lambda| \varepsilon$, then $\sigma \equiv 0$, i.e., $g$ has constant curvature $\lambda$.

In §3 we will prove the following $L^\frac{n}{2}$-version of Theorem 3.

Theorem 4. Given $n$, $D > 0$, and $\lambda < 0$, there is a small constant $\varepsilon = \varepsilon(n, \sqrt{-\lambda}D) > 0$ such that if an Einstein metric $g$, with $\lambda < 0$, on a closed $n$-manifold satisfies $\text{dia}(M) \leq D$ and

$$\int \sigma^\frac{n}{2} \leq |\lambda|^\frac{n}{2} \text{vol}(M) \varepsilon,$$

then $\sigma \equiv 0$, i.e., $g$ has constant curvature $\lambda$.

Complete Einstein metrics on noncompact $n$-manifolds with Einstein constant $\lambda < 0$ are still not completely understood. The Sobolev inequalities do not hold on such manifolds. Instead, the Poincaré inequalities hold, which will be used to prove the following

Theorem 5. Let $g$ be a complete Einstein metric on a noncompact simply connected $n$-manifold with $\lambda < 0$. Suppose $n \geq 10$. There is a small constant $\varepsilon = \varepsilon(n) > 0$ such that if

(i) $\sigma \leq |\lambda| \varepsilon$ and

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(ii) for some \( p \in M \),

\[
\lim_{r \to +\infty} e^{-\delta_n |\lambda|^{1/2}} r \int_{B(p, r)} \sigma^2 = 0,
\]

where \( \delta_n = \frac{1}{2} \sqrt{(n - 1)(n - 9)} > 0 \) and \( B(p, r) \) denotes the geodesic ball of radius \( r \) around \( p \), then \( \sigma \equiv 0 \), i.e., \( g \) has constant curvature \( \lambda < 0 \).

The proof of Theorem 5 will be given in §4. The author does not know the case of \( n \leq 9 \).

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2. Closed Einstein manifolds with \( \lambda = 0 \)

In this section we will prove Theorem 2. The argument given here is quite standard and similar to that given in Lemma 2.1 of [A1].

In §1 we have shown that every Ricci-flat metric satisfying (2) for some small \( \epsilon = \epsilon(n) \) is flat. Throughout this section \( M = (M, g) \) always denotes a closed Ricci-flat manifold of dimension \( n \geq 4 \) and \( c_i(n) \)'s denote constants depending only on \( n \). In the case of \( \lambda = 0 \), (1) is equivalent to

\[
\Delta \sigma + c_0(n)\sigma^2 \geq 0
\]

in the sense of distribution. Recall that the following Sobolev inequality holds in \( M \) (cf. [Be] for references):

\[
\|f\|_{2n \over n-2} \leq c_1(n)\text{vol}(M)^{-\frac{1}{2}}[\text{vol}(M)\|\nabla f\|_2 + \|f\|_2]
\]

for every \( f \in C^\infty(M) \).

Multiply (4) by \( \sigma^\alpha \) for \( \alpha \geq 1 \). Integration by parts gives

\[
c_0(n) \int \sigma^{\alpha+2} \geq \frac{4\alpha}{(\alpha + 1)^2} \int |\nabla \sigma^{\alpha+1}|^2 \geq \frac{1}{\alpha} \int |\nabla \sigma^{\alpha+1}|^2.
\]

Taking \( f = \sigma^{\alpha+1} \) in (5), we obtain by (6)

\[
\|\sigma^{\alpha+1}\|_{2n \over n-2} \leq c_2(n)\text{vol}(M)^{-\frac{1}{2}}[\alpha \text{vol}(M)\|\sigma \cdot \sigma^{\alpha+1}\|_1^1 + \|\sigma^{\alpha+1}\|_2].
\]

Taking \( \alpha + 1 = \frac{q}{2} \) in (7) and applying Hölder’s inequality to \( \|\sigma \cdot \sigma^{n/2}\|_1 \), we have

\[
\|\sigma^{\frac{q}{2}}\|_{2n \over n-2} \leq c_3(n)\text{vol}(M)^{-\frac{1}{2}}[\text{vol}(M)\|\sigma\|_1^1\|\sigma^{\frac{q}{2}}\|_2 + \|\sigma\|_4^1].
\]

It follows from (8) that there is a small constant \( \epsilon(n) > 0 \), such that if for some \( \epsilon \leq \epsilon(n) \)

\[
\|\sigma\|_{n \over 2} \leq \text{vol}(M)^{\frac{1}{2}} \text{dia}(M)^{-\frac{n}{2}} \epsilon,
\]

then

\[
\|\sigma\|_{\frac{q}{2}} = \|\sigma^{\frac{q}{2}}\|_{2n \over n-2} \leq c_4(n)\text{vol}(M)^{-\frac{4}{n}}\|\sigma\|_{\frac{q}{2}} \\
\leq c_5(n)\text{vol}(M)^{\frac{1}{2} - \frac{4}{n} \text{dia}(M)^{-1}},
\]

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where $q = \frac{n}{2} \cdot \frac{2n}{n-2}$. For general $\alpha \geq 1$, by Hölder’s inequality, the interpolation inequality, and (10), we have that for all $\theta > 0$

\[(11)\]
\[
\|\sigma \cdot \sigma^{\alpha+1}\|_1 \leq \|\sigma\|^2_2 \|\sigma^{\alpha+1}\|^2_{2q_2}
\leq c_5(n)\text{vol}(M)^{\frac{2}{n-2} - \delta} \text{dia}(M)^{-2} (\theta \|\sigma^{\alpha+1}\|_{\frac{2n}{n-2}} + \theta^{-\frac{n-2}{2}} \|\sigma^{\alpha+1}\|_2)^2.
\]

Thus it follows from (7) and (11) that

\[(12)\]
\[
\|\sigma^{\alpha+1}\|_{\frac{2n}{n-2}} \leq c_6(n)[\alpha^{\frac{1}{2}} \text{vol}(M)^{-\frac{1}{2}} \theta \|\sigma^{\alpha+1}\|_{\frac{2n}{n-2}}
+ (\alpha^{\frac{1}{2}} \text{vol}(M)^{-\frac{1}{2}} \theta^{-\frac{n-2}{2}} + \text{vol}(M)^{-\frac{1}{2}}) \|\sigma^{\alpha+1}\|_2].
\]

Choosing $\theta = \frac{1}{2} c_6(n)^{-1} \alpha^{-\frac{1}{2}} \text{vol}(M)^{\frac{2}{n-2}}$, we obtain by (12)

\[(13)\]
\[
\|\sigma^{\alpha+1}\|_{\frac{2n}{n-2}} \leq c_7(n)\alpha^{\frac{1}{2}} \text{vol}(M)^{-\frac{1}{2}} \|\sigma^{\alpha+1}\|_2.
\]

Let $x = \frac{n}{n-2}$ and $\alpha + 1 = \frac{n}{2} x$, $i \geq 0$. It follows from (13) that

\[
\|\sigma\|_{\frac{2}{x} x^{i+1}} \leq c_8(n) x^{\frac{i}{2}} \text{vol}(M)^{-\frac{x}{2} \frac{1}{x}} \|\sigma\|_{\frac{2}{x} x^i}
\leq c_8(n) x^{\frac{i}{2} + \frac{i}{2} + \frac{1}{2}} x^{\frac{i}{2} + \frac{1}{2}} \text{vol}(M)^{-\frac{1}{2} \frac{x}{2} x^{i+1} + \frac{x}{2} x^{i+1}} \|\sigma\|_{\frac{2}{x} x^i}.
\]

Letting $i \to +\infty$, we obtain

\[
\sigma \leq c_9(n) \text{vol}(M)^{-\frac{2}{n}} \|\sigma\|_2 \leq c_9(n) \text{dia}(M)^{-2} \epsilon,
\]

i.e.,

\[
\sigma \cdot \text{dia}(M)^2 \leq c_9(n) \epsilon.
\]

The last inequality follows from (9). Choosing a smaller $\epsilon$ in (9) if necessary, by the argument in §1, we conclude that $\sigma \equiv 0$, i.e., $g$ is flat. This completes the proof of Theorem 2.

3. Closed Einstein manifolds with $\lambda < 0$

In this section we will only give a sketch of the proof of Theorem 4. The method applied here is very standard and similar to that given in §2. Let $M = (M, g)$ be a closed Einstein $n$-manifold with Einstein constant $\lambda < 0$ and $\text{dia}(M) \leq D$. Throughout this section $c_i(n)$’s always denote positive constants depending only on $n$.

First one has the following Sobolev inequality in $M$ (cf., e.g., [Be] for references): for every $f \in C^\infty(M)$

\[(14)\]
\[
\|f\|_{\frac{2n}{n-2}} \leq c_1(n) C(\lambda|D|)^{-\frac{1}{2}} \text{vol}(M)^{-\frac{1}{2}} (\lambda^{-\frac{1}{2}} \|\nabla f\|_2 + \|f\|_2),
\]

where $C(x)$, $x > 0$, is the unique positive root of the equation

\[
y \int_0^x (\cosh t + y \sinh t)^{n-1} dt = \int_0^x \sin^{n-1} t dt.
\]
Similarly, by (1) and (14) we obtain that there is a constant $e(n) > 0$ if for some $\varepsilon < e(n)C(\sqrt{|\lambda|D})^2$

\begin{equation}
\|\sigma\|_{\frac{n}{2}} \leq |\lambda|\text{vol}(M)^{\frac{1}{2}}e,
\end{equation}

then for $q = \frac{n}{2} \cdot \frac{2n}{n-2}$

\begin{align*}
\|\sigma\|_{\frac{n}{2}} &\leq c_2(n)C(\sqrt{|\lambda|D})^{-\frac{1}{2}}\text{vol}(M)^{-\frac{1}{4n}}\|\sigma\|_{\frac{n}{2}} \\
&\leq c_3(n)C(\sqrt{|\lambda|D})^{2-\frac{1}{2}}|\lambda|\text{vol}(M)^{\frac{1}{2}-\frac{1}{4n}}
\end{align*}

and for $\alpha \geq 1$,

\begin{equation}
\|\sigma^{(n-1)}\|_{\frac{2n}{n-2}} \leq c_4(n)C(\sqrt{|\lambda|D})^{-\frac{1}{4}}\text{vol}(M)^{-\frac{1}{4n}}\|\sigma^{(n-1)}\|_2.
\end{equation}

Then the last argument in §2 carries over to give

\begin{equation}
\sigma \leq c_5(n)C(\sqrt{-\lambda D})^{-2}|\lambda|e.
\end{equation}

Choosing a smaller $\varepsilon$ in (15) if necessary, by Theorem 3 (Ye), one concludes that $\sigma \equiv 0$, i.e., $g$ has constant curvature $\lambda < 0$.

4. PROOF OF THEOREM 5

Let $M = (M, g)$ be a complete $n$-manifold. Denote by $\lambda_1(M, g)$ the first eigenvalue of $M$, defined as

$$
\lambda_1(M, g) = \inf \left\{ \frac{\int |\nabla f|^2}{\int f^2} \right\},
$$

where the infimum is taken over all $f \in C^\infty_0(M)$ with compact support in $M$. It is proved in [M] that if $M$ is simply connected with sectional curvature $K_g \leq -\Lambda^2$ ($\Lambda > 0$),

$$
\lambda_1(M, g) \geq \frac{(n-1)^2}{4}\Lambda^2,
$$

i.e., for every $f \in C^\infty_0(M)$,

\begin{equation}
\frac{1}{4}(n-1)^2\Lambda^2 \int f^2 \leq \int |\nabla f|^2.
\end{equation}

From now on $(M, g)$ always denotes a complete Einstein $n$-manifold with Einstein constant $\lambda < 0$ and $c_i(n)$'s denote positive constants depending only on $n$. Clearly, there is a small constant $e(n) > 0$ such that if for some $\varepsilon \leq e(n)$, $\sigma \leq |\lambda|e$, then the sectional curvature satisfies

\begin{equation}
K_g \leq -(1 - c_1(n)e)|\lambda| < 0.
\end{equation}

By (1) and (17) we have

\begin{equation}
\Delta \sigma + (c_0(n)e + 2(n-1))|\lambda|\sigma \geq 0.
\end{equation}

Multiply (18) by $\sigma \eta^2$, where $\eta$ is a cut off function of compact support in $M$. Integration by parts gives

\begin{equation}
(c_0(n)e + 2(n-1))|\lambda| \int (\sigma \eta)^2 \geq \int |\nabla (\sigma \eta)|^2 - \int |\nabla \eta|^2 \sigma^2.
\end{equation}

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Taking \( f = \eta \sigma \) in (16), by (17) we have
\[
\frac{1}{4} (n - 1)^2 (1 - c_1(n) \varepsilon) |\lambda| \int |\nabla(\sigma \eta)|^2 \leq \int |\nabla(\sigma \eta)|^2 ;
\]
so that by (19)
\[
(20) \quad \int |\nabla \sigma|^2 \sigma^2 \geq \left[ \frac{1}{4} (n - 1)(n - 9) - c_2(n) \varepsilon \right] |\lambda| \int (\sigma \eta)^2 .
\]
Now suppose \( n \geq 10 \). Take \( \delta(n) = \frac{1}{3} \sqrt{(n - 1)(n - 9)} \). One can choose a smaller \( \varepsilon \) in (17) if necessary, such that
\[
\frac{1}{4} (n - 1)(n - 9) - c_2(n) \varepsilon \geq \frac{1}{4} \varepsilon^2 \delta(n)^2 .
\]
Choosing \( \eta(x) = \eta(d(p, x)) \), where
\[
\eta(t) = \begin{cases} 
1 & \text{if } t \leq r , \\
\frac{R - t}{R - r} & \text{if } r \leq t \leq R , \\
0 & \text{if } t \geq R ,
\end{cases}
\]
we obtain by (20)
\[
(21) \quad \frac{1}{(R - r)^2} \int_{B(p, r)} \sigma^2 \geq \frac{1}{4} \varepsilon^2 \delta(n)^2 |\lambda| \int_{B(p, r)} \sigma^2 .
\]
For any \( r_0 > 0 \), take \( r_j = 2\delta(n)^{-1}|\lambda|^{-\frac{1}{2}}j + r_0 , j \geq 0 \). It then follows from (21) that
\[
\int_{B(p, r_j)} \sigma^2 \geq \varepsilon^2 \int_{B(p, r_{j-1})} \sigma^2 \geq e^{2j} \int_{B(p, r_0)} \sigma^2 = e^{\delta(n)|\lambda|^{-\frac{1}{2}}(r_j - r_0)} \int_{B(p, r_0)} \sigma^2 .
\]
Thus it is easy to see that
\[
\int_{B(p, r_0)} \sigma^2 \leq e^{-\delta(n)|\lambda|^{-\frac{1}{2}}(r_j - r_0)} \int_{B(p, r_j)} \sigma^2 .
\]
Letting \( r_j \to +\infty \), by Theorem 5(ii), one obtains \( \sigma \equiv 0 \) on \( B(p, r_0) \). Since \( r_0 \) is arbitrary, one concludes that \( \sigma \equiv 0 \) on \( M \), i.e., \( g \) has constant curvature \( \lambda < 0 \). This completes the proof of Theorem 5.

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