RIGIDITY THEOREMS FOR NONPOSITIVE EINSTEIN METRICS

ZHONGMIN SHEN

(Communicated by Jonathan M. Rosenberg)

Abstract. In this paper we study the following problem: When must a complete Einstein metric $g$ on an $n$-manifold with $\text{Ric} = (n - 1)\lambda g$, $\lambda \leq 0$, be a metric of constant curvature $\lambda$?

1. Introduction and main results

Let $g$ be a complete riemannian metric on an $n$-manifold $M$. Denote by $R$ the curvature tensor of $g$. The Ricci curvature $\text{Ric}$ is then defined as

$$\text{Ric}(x, y) = \sum_{i=1}^{n} g(R(x, e_i) e_i, y), \quad x, y \in T_p M,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $T_p M$. The metric $g$ is said to be Einstein if the Ricci curvature is constant, i.e.,

$$\text{Ric} = (n - 1)\lambda g$$

for some constant $\lambda$. $\lambda$ is called the Einstein constant of $g$. It is clear that in dimension three the metric $g$ is Einstein with $\text{Ric} = (n - 1)\lambda g$ if and only if it has constant curvature $\lambda$, i.e.,

$$R(x, y)z = \lambda(g(y, z)x - g(x, z)y), \quad x, y, z \in T_p M.$$

In higher dimensions, this is not the case. One may ask if an Einstein metric has constant curvature whenever it has almost constant curvature in a certain sense. From now on we always assume that $g$ is a complete Einstein metric with Einstein constant $\lambda$. It is natural to consider the new tensor $\hat{R}$, defined

Received by the editors January 8, 1991 and, in revised form, March 19, 1991.
1991 Mathematics Subject Classification. Primary 53C25; Secondary 53C20.
Key words and phrases. Einstein metrics, rigidity, Sobolev inequality, the first eigenvalue, diameter, volume.
Research at MSRI supported in part by NSF Grant DMS-8505550.
by
\[ \hat{R}(x, y) z = R(x, y) z - \lambda(g(y, z)x - g(x, z)y), \quad x, y, z \in T_p M. \]

Denote by \( \sigma \) the pointwise norm of \( \hat{R} \), defined by
\[ \sigma = \sqrt{\sum_{ijkl} g(\hat{R}(e_i, e_j)e_k, e_l)^2}, \]
where \( \{e_1, \ldots, e_n\} \) is an orthonormal basis for \( T_p M \). By a formula in [H], one can easily show that \( \sigma \) satisfies
\[ (1) \quad \Delta \sigma + c_0(n)\sigma^2 - 2(n-1)\lambda \sigma \geq 0 \]
in the sense of distribution, where \( c_0(n) \) is a positive constant depending only on \( n \) and \( \Delta \) denotes the Laplace-Beltrami operator (in \( \mathbb{R}^n \), \( \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \)).

In the case of \( \lambda > 0 \), Myers's Theorem (cf., e.g., [CE]) tells us that the manifold is closed. Thus by integrating (1), one obtains

**Theorem 1** (Berger [B]). Given \( n \), there is a small constant \( \varepsilon = \varepsilon(n) > 0 \) depending only on \( n \) such that if a complete Einstein metric \( g \), with \( \lambda > 0 \), on an \( n \)-manifold satisfies \( \sigma \leq \lambda \varepsilon \), then \( \sigma \equiv 0 \), i.e., \( g \) has constant curvature \( \lambda \).

In [S] the author gives an \( L^\frac{2}{3} \)-version of Theorem 1, which says that if
\[ \int \sigma^{\frac{2}{3}} \leq \lambda^{\frac{2}{3}} \text{vol}(M) \varepsilon \]
for some small \( \varepsilon = \varepsilon(n) > 0 \), depending only on \( n \), where \( \text{vol}(M) \) denotes the volume of \( (M, g) \), then \( \sigma \equiv 0 \).

In the case of \( \lambda \leq 0 \), the manifold can be compact or noncompact. First let us consider the case \( \lambda = 0 \). In this case, the following fact is known: There is a small constant \( \varepsilon = \varepsilon(n) > 0 \) depending only on \( n \), if a Ricci-flat metric \( g \) on a closed \( n \)-manifold satisfies
\[ (2) \quad \sigma \cdot \text{dia}(M)^2 \leq \varepsilon, \]
where \( \text{dia}(M) \) denotes the diameter of \( g \), then \( \sigma \equiv 0 \), i.e., \( g \) is flat. The proof of this fact is trivial. By a theorem of Gromov [G], any almost flat manifold is aspherical, i.e., its universal cover is diffeomorphic to \( \mathbb{R}^n \). Thus for a sufficiently small \( \varepsilon = \varepsilon(n) \), (2) implies that the universal cover \( \tilde{M} \) is diffeomorphic to \( \mathbb{R}^n \). On the other hand, by the Cheeger-Gromoll's Splitting Theorem (cf. [CG]), \( \tilde{M} \) with the induced metric \( \tilde{g} \) is isometric to a riemannian product \( N \times \mathbb{R}^k \) for some closed riemannian manifold \( N \). Thus \( N \) must be a point and \( \tilde{g} \) is flat. Therefore \( g \) is flat. This argument in fact shows that all nonnegatively Ricci-curved aspherical manifolds are flat. By a theorem of Fukaya-Yamaguchi [FY], if \( \text{dia}(M) \leq D \), then condition (2) can be replaced by \( -1 \leq K_g \leq \varepsilon \) for a small number \( \varepsilon = \varepsilon(n, D) > 0 \), where \( K_g \) denotes the sectional curvature of \( g \).

In §2 we will prove an \( L^\frac{2}{3} \)-version of the above fact, that is,
Theorem 2. Given $n$, there is a small constant $\varepsilon = \varepsilon(n) > 0$ depending only on $n$ such that if a Ricci-flat metric $g$ on a closed $n$-manifold satisfies
\[ \int \sigma^\frac{n}{n+1} \leq \frac{\text{vol}(M)}{\text{dia}(M)^n} \varepsilon, \]
then $\sigma \equiv 0$, i.e., $g$ is flat.

For complete Ricci-flat metrics on noncompact $n$-manifolds, some rigidity phenomena have been discovered (cf. [A2, Ba, S], etc.). Roughly speaking, if a complete Ricci-flat metric $g$ has sufficiently small total curvature, i.e., there is a small $\varepsilon = \varepsilon(n) > 0$ such that if
\[ \int \sigma^\frac{n}{n+1} \nu_M^{n+1} \varepsilon \]
where
\[ \nu_M := \lim_{r \to +\infty} \frac{\text{vol}(B(p, r))}{\sigma_n r^n} > 0, \]
where $\sigma_n$ denotes the volume of the unit ball in $\mathbb{R}^n$, then $\sigma \equiv 0$. It is worth mentioning that the result of Anderson [A2] does not require (3), but requires that $\nu_M \geq 1 - \varepsilon$ for a small $\varepsilon = \varepsilon(n) > 0$.

Now let us consider the case of $\lambda < 0$. The following theorem is first proved by Ye [Y, Theorem 2].

Theorem 3 ([Y]). Given $n$, $D > 0$, and $\lambda < 0$, there is a small constant $\varepsilon = \varepsilon(n, \sqrt{-\lambda}D) > 0$ such that if an Einstein metric $g$, with $\lambda < 0$, on a closed $n$-manifold satisfies $\text{dia}(M) \leq D$ and $\sigma \leq |\lambda| \varepsilon$, then $\sigma \equiv 0$, i.e., $g$ has constant curvature $\lambda$.

In §3 we will prove the following $L^\frac{n}{n+1}$-version of Theorem 3.

Theorem 4. Given $n$, $D > 0$, and $\lambda < 0$, there is a small constant $\varepsilon = \varepsilon(n, \sqrt{-\lambda}D) > 0$ such that if an Einstein metric $g$, with $\lambda < 0$, on a closed $n$-manifold satisfies $\text{dia}(M) \leq D$ and
\[ \int \sigma^\frac{n}{n+1} \leq |\lambda| \frac{\text{vol}(M)}{\sigma_n} \varepsilon, \]
then $\sigma \equiv 0$, i.e., $g$ has constant curvature $\lambda$.

Complete Einstein metrics on noncompact $n$-manifolds with Einstein constant $\lambda < 0$ are still not completely understood. The Sobolev inequalities do not hold on such manifolds. Instead, the Poincaré inequalities hold, which will be used to prove the following

Theorem 5. Let $g$ be a complete Einstein metric on a noncompact simply connected $n$-manifold with $\lambda < 0$. Suppose $n \geq 10$. There is a small constant $\varepsilon = \varepsilon(n) > 0$ such that if
(i) $\sigma \leq |\lambda| \varepsilon$ and
(ii) for some \( p \in M \),
\[
\lim_{r \to +\infty} e^{-\delta_n |\lambda|^{\frac{1}{n}}} \int_{B(p, r)} \sigma^2 = 0,
\]
where \( \delta_n = \frac{1}{2} \sqrt{(n-1)(n-9)} > 0 \) and \( B(p, r) \) denotes the geodesic ball of radius \( r \) around \( p \), then \( \sigma \equiv 0 \), i.e., \( g \) has constant curvature \( \lambda < 0 \).

The proof of Theorem 5 will be given in §4. The author does not know the case of \( n \leq 9 \).

The author would like to thank S. Bando for bringing the problem of Theorem 5 to his attention. Thanks also to M. Anderson for many helpful discussions.

2. Closed Einstein manifolds with \( \lambda = 0 \)

In this section we will prove Theorem 2. The argument given here is quite standard and similar to that given in Lemma 2.1 of [A1].

In §1 we have shown that every Ricci-flat metric satisfying (2) for some small \( \epsilon = \epsilon(n) \) is flat. Throughout this section \( M = (M, g) \) always denotes a closed Ricci-flat manifold of dimension \( n \geq 4 \) and \( c_i(n) \)'s denote constants depending only on \( n \). In the case of \( \lambda = 0 \), (1) is equivalent to
\[
\Delta \sigma + c_0(n) \sigma^2 \geq 0
\]
in the sense of distribution. Recall that the following Sobolev inequality holds in \( M \) (cf. [Be] for references):
\[
\|f\|_{\frac{2n}{n-2}} \leq c_1(n) \text{vol}(M)^{-\frac{1}{2}} [\text{dia}(M)] \|\nabla f\|_2 + \|f\|_2
\]
for every \( f \in C^\infty(M) \).

Multiply (4) by \( \sigma^\alpha \) for \( \alpha \geq 1 \). Integration by parts gives
\[
c_0(n) \int \sigma^{\alpha+2} \geq \frac{4\alpha}{(\alpha+1)^2} \int |\nabla \sigma^{\frac{\alpha+1}{2}}|^2 \geq \frac{1}{\alpha} \int |\nabla \sigma^{\frac{\alpha+1}{2}}|^2.
\]
Taking \( f = \sigma^{\frac{\alpha+1}{2}} \) in (5), we obtain by (6)
\[
\|\sigma^{\frac{\alpha+1}{2}}\|_{\frac{2n}{n-2}} \leq c_2(n) \text{vol}(M)^{-\frac{1}{2}} [\alpha \frac{1}{2} \text{dia}(M)] \|\sigma \cdot \sigma^{\alpha+1}\|_{\frac{1}{2}} + \|\sigma^{\frac{\alpha+1}{2}}\|_2.
\]
Taking \( \alpha + 1 = \frac{2}{\beta} \) in (7) and applying Hölder’s inequality to \( \|\sigma \cdot \sigma^{n/2}\|_1 \), we have
\[
\|\sigma^{\frac{n}{2}}\|_{\frac{2n}{n-2}} \leq c_3(n) \text{vol}(M)^{-\frac{1}{2}} [\text{dia}(M)] \|\sigma\|_{\frac{1}{2}} \|\sigma^{\frac{n}{2}}\|_{\frac{2n}{n-2}} + \|\sigma^{\frac{n}{2}}\|_2.
\]
It follows from (8) that there is a small constant \( \epsilon(n) \) such that if for some \( \epsilon \leq \epsilon(n) \)
\[
\|\sigma\|_{\frac{1}{2}} \leq \frac{\text{vol}(M)^{\frac{1}{2}}}{\text{dia}(M)^{\frac{1}{2}}} \epsilon,
\]
then
\[
\|\sigma\|_{\frac{3}{2}} = \|\sigma^{\frac{3}{2}}\|_{\frac{2n}{n-2}} \leq c_4(n) \text{vol}(M)^{-\frac{3}{2}} \|\sigma\|_{\frac{3}{2}} \leq c_5(n) \text{vol}(M)^{\frac{1}{2}} \text{dia}(M)^{-\frac{1}{2}},
\]
for some positive constant \( c_4(n) \) and \( c_5(n) \).
where \( q = \frac{n}{2} \cdot \frac{2n}{n-2} \). For general \( \alpha \geq 1 \), by Hölder’s inequality, the interpolation inequality, and (10), we have that for all \( \theta > 0 \)

\[
(11) \quad \|\sigma \cdot \sigma^{\alpha+1}\|_1 \leq \|\sigma\|_\frac{2}{\alpha} \|\sigma^{\frac{q+1}{2}}\|_2^{\frac{2}{q+2}}
\]

\[
\leq c_6(n) \text{vol}(M)^{\frac{2}{n+2} - \frac{3}{2}} \text{dia}(M)^{-2} (\theta \|\sigma^{\frac{1}{2}}\|_2 + \theta^{-\frac{n-3}{2}} \|\sigma^{\frac{q+1}{2}}\|_2)^2.
\]

Thus it follows from (7) and (11) that

\[
(12) \quad \|\sigma^{\frac{q+1}{2}}\|_{\frac{2n}{n-2}} \leq c_6(n) [\alpha^{\frac{1}{2}} \text{vol}(M)^{-\frac{1}{2}} \theta \|\sigma^{\frac{q+1}{2}}\|_{\frac{2n}{n-2}} + (\alpha^{\frac{1}{2}} \text{vol}(M)^{-\frac{1}{2}} \theta^{-\frac{n-3}{2}} + \text{vol}(M)^{-\frac{1}{2}}) \|\sigma^{\frac{q+1}{2}}\|_2].
\]

Choosing \( \theta = \frac{1}{2} c_6(n)^{-1} \alpha^{-\frac{1}{2}} \text{vol}(M)^{\frac{2}{n-2}} \), we obtain by (12)

\[
(13) \quad \|\sigma^{\frac{q+1}{2}}\|_{\frac{2n}{n-2}} \leq c_7(n) \alpha^{\frac{q}{2}} \text{vol}(M)^{-\frac{1}{2}} \|\sigma^{\frac{q+1}{2}}\|_2.
\]

Let \( \chi = \frac{n}{n-2} \) and \( \alpha + 1 = \frac{n}{2} \chi \), \( i \geq 0 \). It follows from (13) that

\[
\|\sigma\|_{\frac{q}{2} \chi^{i+1}} \leq c_8(n) \frac{1}{2} \chi \chi^i \text{vol}(M)^{-\frac{1}{2}} \frac{1}{2} \|\sigma\|_{\frac{q}{2} \chi^{i}}
\]

\[
\leq c_8(n) \frac{1}{\sqrt{\chi}} \cdots \frac{1}{\sqrt{\chi}} \frac{1}{\sqrt{\chi}} \chi^{\frac{1}{2} + \cdots + \frac{1}{2}} \text{vol}(M)^{-\frac{1}{2}} \frac{1}{2} \frac{1}{2} \cdots \frac{1}{2} \|\sigma\|_{\frac{q}{2}}.
\]

Letting \( i \to +\infty \), we obtain

\[
\sigma \leq c_9(n) \text{vol}(M)^{-\frac{1}{2}} \|\sigma\|_{\frac{q}{2}} \leq c_9(n) \text{dia}(M)^{-2} \epsilon,
\]

i.e.,

\[
\sigma \cdot \text{dia}(M)^2 \leq c_9(n) \epsilon.
\]

The last inequality follows from (9). Choosing a smaller \( \epsilon \) in (9) if necessary, by the argument in §1, we conclude that \( \sigma \equiv 0 \), i.e., \( g \) is flat. This completes the proof of Theorem 2.

3. Closed Einstein manifolds with \( \lambda < 0 \)

In this section we will only give a sketch of the proof of Theorem 4. The method applied here is very standard and similar to that given in §2. Let \( M = (M, g) \) be a closed Einstein \( n \)-manifold with Einstein constant \( \lambda < 0 \) and \( \text{dia}(M) \leq D \). Throughout this section \( c_1(n) \)’s always denote positive constants depending only on \( n \).

First one has the following Sobolev inequality in \( M \) (cf., e.g., [Be] for references): for every \( f \in C^\infty(M) \)

\[
(14) \quad \|f\|_{\frac{2n}{n-2}} \leq c_1(n) C(\sqrt{|\lambda|} D)^{-\frac{1}{2}} \text{vol}(M)^{-\frac{1}{2}} (|\lambda|^{-\frac{1}{2}} \|\nabla f\|_2 + \|f\|_2),
\]

where \( C(x), x > 0, \) is the unique positive root of the equation

\[
y \int_0^x (\cosh t + y \sinh t)^{n-1} dt = \int_0^{\pi} \sin^{n-1} t dt.
\]
Similarly, by (1) and (14) we obtain that there is a constant \( \varepsilon(n) > 0 \) if for some \( \varepsilon < \varepsilon(n) \)
\[
\| \sigma \|_{q} \leq |\lambda| \text{vol}(M)^{\frac{1}{2}} \varepsilon
\]
then for \( q = \frac{n}{2} \cdot \frac{2n}{n-2} \)
\[
\| \sigma \|_{q} \leq c_{2}(n)C(\sqrt{\lambda}|D|)^{-\frac{n}{2}} \text{vol}(M)^{-\frac{n}{4}} \| \sigma \|_{q} \leq c_{3}(n)C(\sqrt{\lambda}|D|)^{2-\frac{n}{2}} |\lambda| \text{vol}(M)^{\frac{n}{2}-\frac{n}{2} \varepsilon}
\]
and for \( \alpha \geq 1 \),
\[
\| \sigma^{\frac{\alpha}{2}} \|_{2} \leq c_{4}(n)C(\sqrt{\lambda}|D|)^{-1} \alpha^{\frac{n}{2}} \text{vol}(M)^{-\frac{1}{2}} \| \sigma^{\frac{\alpha}{2}} \|_{2}.
\]
Then the last argument in §2 carries over to give
\[
\sigma \leq c_{5}(n)C(\sqrt{\lambda}|D|)^{-2}|\lambda| \varepsilon.
\]
Choosing a smaller \( \varepsilon \) in (15) if necessary, by Theorem 3 (Ye), one concludes that \( \sigma \equiv 0 \), i.e., \( g \) has constant curvature \( \lambda < 0 \).

4. PROOF OF THEOREM 5

Let \( M = (M, g) \) be a complete \( n \)-manifold. Denote by \( \lambda_{1}(M, g) \) the first eigenvalue of \( M \), defined as
\[
\lambda_{1}(M, g) = \inf \frac{\int |\nabla f|^{2}}{\int f^{2}}
\]
where the infimum is taken over all \( f \in C_{0}^{\infty}(M) \) with compact support in \( M \). It is proved in [M] that if \( M \) is simply connected with sectional curvature \( K_{g} \leq -\Lambda^{2} (\Lambda > 0) \),
\[
\lambda_{1}(M, g) \geq \frac{(n-1)^{2}}{4} \Lambda^{2},
\]
i.e., for every \( f \in C_{0}^{\infty}(M) \),
\[
\frac{1}{4} (n - 1)^{2} \Lambda^{2} \int f^{2} \leq \int |\nabla f|^{2}.
\]
From now on \( (M, g) \) always denotes a complete Einstein \( n \)-manifold with Einstein constant \( \lambda < 0 \) and \( c_{i}(n) \)'s denote positive constants depending only on \( n \). Clearly, there is a small constant \( \varepsilon(n) > 0 \) such that if for some \( \varepsilon \leq \varepsilon(n) \), \( \sigma \leq |\lambda| \varepsilon \), then the sectional curvature satisfies
\[
K_{g} \leq -(1 - c_{1}(n) \varepsilon)|\lambda| < 0.
\]
By (1) and (17) we have
\[
(18) \quad \Delta \sigma + (c_{0}(n) \varepsilon + 2(n - 1)) |\lambda| \sigma \geq 0.
\]
Multiply (18) by \( \sigma \eta^{2} \), where \( \eta \) is a cut off function of compact support in \( M \). Integration by parts gives
\[
(19) \quad (c_{0}(n) \varepsilon + 2(n - 1)) |\lambda| \int (\sigma \eta)^{2} \geq \int |\nabla (\sigma \eta)|^{2} - \int |\nabla \eta|^{2} \sigma^{2}.
\]
Taking \( f = \eta \sigma \) in (16), by (17) we have
\[
\frac{1}{4}(n-1)^2(1-c_1(n)e)|\lambda| \int (\sigma \eta)^2 \leq \int |\nabla (\sigma \eta)|^2;
\]
so that by (19)
\[
(20) \quad \int |\nabla \eta|^2 \sigma^2 \geq \left[ \frac{1}{4}(n-1)(n-9) - c_2(n)e \right] |\lambda| \int (\sigma \eta)^2.
\]
Now suppose \( n \geq 10 \). Take \( \delta(n) = \frac{1}{3}\sqrt{(n-1)(n-9)} \). One can choose a smaller \( \varepsilon \) in (17) if necessary, such that
\[
\frac{1}{4}(n-1)(n-9) - c_2(n)e \geq \frac{1}{4}e^2\delta(n)^2.
\]
Choosing \( \eta(x) = \eta(d(p,x)) \), where
\[
\eta(t) = \begin{cases} 
1 & \text{if } t \leq r, \\
\frac{R-t}{R-r} & \text{if } r \leq t \leq R, \\
0 & \text{if } t \geq R,
\end{cases}
\]
we obtain by (20)
\[
(21) \quad \frac{1}{(R-r)^2} \int_{B(p,r)} \sigma^2 \geq \frac{1}{4}e^2\delta(n)^2|\lambda| \int_{B(p,r)} \sigma^2.
\]
For any \( r_0 > 0 \), take \( r_j = 2\delta(n)^{-1}|\lambda|^{-\frac{1}{2}}j + r_0, \ j \geq 0 \). It then follows from (21) that
\[
\int_{B(p,r_j)} \sigma^2 \geq e^2 \int_{B(p,r_{j-1})} \sigma^2 \geq e^{2j} \int_{B(p,r_0)} \sigma^2 = e^{\delta(n)|\lambda|^{\frac{1}{2}}(r_j-r_0)} \int_{B(p,r_0)} \sigma^2.
\]
Thus it is easy to see that
\[
\int_{B(p,r_0)} \sigma^2 \leq e^{-\delta(n)|\lambda|^{\frac{1}{2}}(r_j-r_0)} \int_{B(p,r_j)} \sigma^2.
\]
Letting \( r_j \to +\infty \), by Theorem 5(ii), one obtains \( \sigma \equiv 0 \) on \( B(p,r_0) \). Since \( r_0 \) is arbitrary, one concludes that \( \sigma \equiv 0 \) on \( M \), i.e., \( g \) has constant curvature \( \lambda < 0 \). This completes the proof of Theorem 5.

**References**


**Mathematical Sciences Research Institute, Berkeley, California 94720**

*Current address*: Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109-1003

*e-mail*: zhongmin@math.lsa.umich.edu