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EPIMORPHISMS AND MONOMORPHISMS IN HOMOTOPY

JERZY DYDAK

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ABSTRACT. The main result of this note is the following:

Theorem A. *If* $f: X \to Y$ *is an epimorphism of* $\mathscr{H}\mathscr{C}\mathscr{W}^*$, *the homotopy category of pointed path-connected CW-spaces, and* $\pi_1(f): \pi_1(X) \to \pi_1(Y)$ *is a monomorphism, then* $\tilde{f}: \tilde{X} \to \tilde{Y}$ *is an epimorphism of* $\mathscr{H}\mathscr{C}\mathscr{W}^*$.

As a straightforward consequence the following results of Dyer-Roitberg (Topology Appl. (to appear)) is derived:

Theorem B. *A map* $f: X \to Y$ *is an equivalence in* $\mathscr{H}\mathscr{C}\mathscr{W}^*$, *the homotopy category of pointed path-connected CW-spaces, iff it is both an epimorphism and a monomorphism in* $\mathscr{H}\mathscr{C}\mathscr{W}^*$.

Recall that f is a monomorphism (epimorphism) in $\mathscr{H}\mathscr{C}\mathscr{W}^*$ if given $\alpha, \beta: Z \to X$ $(\alpha, \beta: Y \to Z)$, $f \circ \alpha \approx f \circ \beta$ implies $\alpha \approx \beta$ $(\alpha \circ f \approx \beta \circ f$ implies $\alpha \approx \beta)$.

Lemma 1. *An inclusion* $f: X \to Y$ *is an epimorphism of* $\mathscr{H}\mathscr{C}\mathscr{W}^*$ *iff there is a map* $H: Y \times [-1, 1] \to (Y \times \{-1, 1\} \cup X \times [-1, 1])$ *such that*

$$H|Y \times \{-1, 1\} \cup \{*\} \times [-1, 1]:$$
$$Y \times \{-1, 1\} \cup \{*\} \times [-1, 1] \to Y \times \{-1, 1\} \cup X \times [-1, 1]$$

is the inclusion.

Proof. Given two maps $g, h: Y \to Z$ such that $g \circ f \approx h \circ f$, there is $G: Y \times \{-1, 1\} \cup X \times [-1, 1] \to Z$ such that $G|Y \times \{-1\} = g$, $G|Y \times \{1\} = h$, and $G|\{*\} \times I = \text{const}$. Now, $G \circ H: Y \times [-1, 1] \to Z$ is a homotopy joining g and h, i.e., f is an epimorphism of $\mathscr{H}\mathscr{C}\mathscr{W}^*$.

Suppose f is an epimorphism of $\mathscr{H}\mathscr{C}\mathscr{W}^*$. Consider the two inclusion $j_i: Y \to (Y \times \{-1, 1\} \cup X \times [-1, 1])/\{*\} \times [-1, 1]$, $j_i(y) = (y, i)$, $i = -1, 1$, and notice that $j_{-1} \circ f \approx j_1 \circ f$. Hence, $j_{-1} \approx j_1$; i.e., there is $H': Y \times [-1, 1] \to (Y \times \{-1, 1\} \cup X \times [-1, 1])/\{*\} \times [-1, 1]$ such that $H'(y, i) = (y, i)$ for $(y, i) \in Y \times \{-1, 1\}$. Since the projection $Y \times \{-1, 1\} \cup X \times [-1, 1] \to (Y \times \{-1, 1\} \cup X \times [-1, 1])/\{*\} \times [-1, 1]$ is a homotopy equivalence, Lemma 1

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follows (the map $Y\times\{-1,1\}\cup\{*\}\times[-1,1]\to(Y\times\{-1,1\}\cup X\times[-1,1])/\{*\}\times[-1,1]$ extends over $Y\times[-1,1]$, so the inclusion $Y\times\{-1,1\}\cup\{*\}\times[-1,1]\to Y\times\{1,1\}\cup X\times[-1,1]$ extends over $Y\times[-1,1]$).

Theorem A. *If $f\colon X\to Y$ is an epimorphism of $\mathscr{H}\mathscr{C}\mathscr{W}^*$, the homotopy category of pointed path-connected CW-spaces and if $\pi_1(f)\colon\pi_1(X)\to\pi_1(Y)$ is a monomorphism, then $\tilde{f}\colon\widetilde{X}\to\widetilde{Y}$ is an epimorphism of $\mathscr{H}\mathscr{C}\mathscr{W}^*$.*

Proof. For simplicity, assume $X\subset Y$ and f is the inclusion (replace f by the inclusion $X\to M(f)$ from X to the reduced mapping cylinder of f). Let $\pi\colon\widetilde{Y}\to Y$ be the projection from the universal cover of Y to Y. Choose a base point $*$ in $\pi^{-1}(*)$. Notice that $p=\pi\times\mathrm{id}\colon\widetilde{Y}\times[-1,1]\to Y\times[-1,1]$ is the universal covering projection. If $\pi_1(f)$ is an isomorphism, then $\pi^{-1}(X)$ is the universal cover $\widetilde{X}$ of X and $p^{-1}(Y\times\{-1,1\}\cup X\times[-1,1])=\widetilde{Y}\times\{-1,1\}\cup\widetilde{X}\times[-1,1]$ is the universal cover of $Y\times\{-1,1\}\cup X\times[-1,1]$ as the inclusion $Y\times\{-1,1\}\cup X\times[-1,1]\to Y\times[-1,1]$ induces an isomorphism of fundamental groups. Choose the lift $\widetilde{H}\colon\widetilde{Y}\times[-1,1]\to\widetilde{Y}\times\{-1,1\}\cup\widetilde{X}\times[0,1]$ of H with $\widetilde{H}(*\times\{-1\})=\{*\}$. Clearly, $\widetilde{H}|\widetilde{Y}\times\{-1,1\}=\mathrm{id}$, which proves that $\tilde{f}$ is an epimorphism of $\mathscr{H}\mathscr{C}\mathscr{W}^*$.

Lemma 2. *If $f\colon X\to Y$ is an epimorphism of $\mathscr{H}\mathscr{C}\mathscr{W}^*$, then*

(a) *$\pi_1(f)\colon\pi_1(X)\to\pi_1(Y)$ is an epimorphism,*
(b) *$H_k(f)\colon H_k(X)\to H_k(Y)$ is an epimorphism for all $k\geq 1$,*

Proof. For simplicity, assume $X\subset Y$ and f is the inclusion (replace f by the inclusion $X\to M(f)$ from X to the reduced mapping cylinder of f). Notice that $\pi_1((Y\times\{-1,1\}\cup X\times[-1,1])/\{*\}\times[-1,1])$ is the amalgamated product of two copies of $\pi_1(Y)$, with two copies of $\mathrm{im}(\pi_1(X)\to\pi_1(Y))$ identified. The existence of $H\colon Y\times[-1,1]\to(Y\times\{-1,1\}\cup X\times[-1,1])/\{*\}\times[-1,1]$ as in Lemma 1 means that

$$\pi_1(Y\times\{-1\})=\pi_1(Y\times\{1\})$$

in $\pi_1((Y\times\{-1,1\}\cup X\times[-1,1])/\{*\}\times[-1,1])$, which is possible only if $\pi_1(f)$ is an epimorphism. This proves (a).

Consider the projection $\pi\colon Y\to Y/X$ and the trivial map $c\colon Y\to Y/X$. Since $\pi\circ f=c\circ f$, it follows that $\pi\approx c$. From the homology sequence $H_k(X)\to H_k(Y)\to H_k(Y/X)\to\cdots$ we deduce (b).

Corollary. *If $f\colon X\to Y$ is an epimorphism of $\mathscr{H}\mathscr{C}\mathscr{W}^*$ and $\pi_k(f)$ is a monomorphism for all $k\geq 1$, then f is an equivalence of $\mathscr{H}\mathscr{C}\mathscr{W}^*$.*

Proof. It suffices to show that $\pi_k(f)$ is an isomorphism for all $k\geq 1$. By the above result $\pi_1(f)$ is an isomorphism and we may assume (by switching to universal covers) that X and Y are simply connected. Suppose $\pi_n(Y,X)=0$ for all $n\leq m$. Then, $\pi_{m+1}(Y,X)\approx H_{m+1}(Y,X)$ and from the diagram

$$\begin{array}{ccccccc}
\pi_{m+1}(X) & \xrightarrow{\text{mono}} & \pi_{m+1}(Y) & \longrightarrow & \pi_{m+1}(Y,X) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
H_{m+1}(X) & \xrightarrow{\text{epi}} & H_{m+1}(Y) & \xrightarrow{0} & H_{m+1}(Y,X) & \longrightarrow & H_m(X)
\end{array}$$

we conclude that $\pi_{m+1}(Y,X)\to H_{m+1}(Y,X)$ is trivial. Thus, $\pi_n(Y,X)=0$ for all $n\geq 1$ and f is an equivalence.

Now, the following result of Dyer and Roitberg [DR] (see [HR, R_1, R_2] for special versions of it) is a direct consequence of the Corollary:

Theorem B. *A map $f\colon X \to Y$ is an equivalence in $\mathcal{HCW}^*$, the homotopy category of pointed path-connected CW-spaces, iff it is both an epimorphism and a monomorphism in $\mathcal{HCW}^*$.*

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Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996

E-mail address: dydak@utkvx.bitnet