PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 4, December 1992

CHARACTERIZATIONS OF CERTAIN CLASSES OF HEREDITARY C^*-SUBALGEBRAS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. This paper characterizes the class of full hereditary C^*-subalgebras and the class of hereditary C^*-subalgebras that generate essential ideals in a given C^*-algebra in terms of a certain projection of norm one from the enveloping von Neumann algebra of the C^*-algebra onto the enveloping von Neumann algebra of a hereditary C^*-subalgebra. For a C^*-dynamical system (A, G, α), it is also shown that an α-invariant C^*-subalgebra B of A is a hereditary C^*-subalgebra belonging to either of the above classes if and only if the reduced C^*-crossed product $B \times_{\alpha r} G$ is a hereditary C^*-subalgebra, of the reduced C^*-crossed product $A \times_{\alpha r} G$, belonging to the same class as B. Furthermore similar results for C^*-crossed products are also observed.

1. INTRODUCTION

Let A be a C^*-algebra and let B be a C^*-subalgebra of A. We denote by A^{**} the enveloping von Neumann algebra of A, which is identified with the second dual of A. Then the enveloping von Neumann algebra B^{**} of B is identified with the strong closure of B in A^{**} (e.g., [4, 3.7.9]). In [2, Theorem 2.2], the author showed that B is a hereditary C^*-subalgebra of A if and only if there exists a projection of norm one Q from A^{**} onto B^{**} such that $\overline{\varphi}\circ Q = \overline{\varphi}$ for every state φ on B where $\overline{\varphi}$ denotes an extension of a state φ of B to a state of A. By the way, there are a few classes of hereditary C^*-subalgebras playing an important role in various branches of the theory of C^*-algebras. In this paper, we shall deal with full hereditary C^*-subalgebras and the hereditary C^*-subalgebras that generate essential ideals in a given C^*-algebra. Recall that a hereditary C^*-subalgebra is said to be *full* if it is not contained in any proper closed two-sided ideal of the whole C^*-algebra. Full hereditary C^*-subalgebras are very important in the theory of Morita equivalence for C^*-algebras and related topics (e.g., [5] and the references cited therein). On the other hand, the class of the hereditary C^*-subalgebras that generate essential ideals is indispensable in spectral theory of automorphism groups of C^*-algebras (e.g., [4, 8.8]). In §2, we shall give characterizations of such classes of those hereditary C^*-subalgebras in terms of the projection of norm one Q above. More precisely, we show that B is a hereditary C^*-subalgebra of A that generates an

Received by the editors April 8, 1991.
1991 *Mathematics Subject Classification*. Primary 46L05; Secondary 46L55.

essential ideal of A if and only if the projection of norm one Q from A^{**} onto B^{**} is injective on the set of central covers, in A^{**}, of positive elements in A, and that B is a full hereditary C^*-subalgebra of A if and only if the projection of norm one Q from A^{**} onto B^{**} is injective on the center of A^{**}.

Let (A, G, α) be a C^*-dynamical system and denote by $A \times_\alpha G$ (resp. $A \times_{\alpha r} G$) the C^*-crossed product (resp. reduced C^*-crossed product) of A by G. Let B be an α-invariant C^*-subalgebra of A. In §3, it is shown that B is a full hereditary C^*-subalgebra of A if and only if $B \times_\alpha G$ (resp. $B \times_{\alpha r} G$) is a full hereditary C^*-subalgebra of $A \times_\alpha G$ (resp. $A \times_{\alpha r} G$). And we shall also show that B is a hereditary C^*-subalgebra of A that generates an essential ideal of A if and only if $B \times_{\alpha r} G$ is a hereditary C^*-subalgebra of $A \times_{\alpha r} G$ that generates an essential ideal of $A \times_{\alpha r} G$ and that these conditions are necessary conditions for $B \times_\alpha G$ to be a hereditary C^*-subalgebra of $A \times_\alpha G$ that generates an essential ideal of $A \times_\alpha G$.

2. Hereditary C^*-subalgebras

Let A be a C^*-algebra and let B be a C^*-subalgebra of A. Throughout this paper, we denote by $I(B)$ the closed two-sided ideal of A generated by B, and the identity of the von Neumann subalgebra B^{**} of A^{**} is always denoted by p, which is a projection of A^{**}. For each selfadjoint element $x \in A^{**}$, the infimum $c(x)$ of all selfadjoint elements in the center of A^{**} majorizing x is called the *central cover* (or *central support*) of x in A^{**} (see [4, 2.6] for basic results). We denote by $C(A_+)$ the set consisting only of central covers (in A^{**}) of positive elements in A. Let φ be a state on B and denote by $\overline{\varphi}$ an extension of φ to a state on A. As already mentioned in the introduction, our starting point is the result that B is a hereditary C^*-subalgebra of A if and only if there exists a projection of norm one Q from A^{**} onto B^{**} such that $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on B. And the key point is that Q is uniquely determined as the form $Q(\cdot) = p \cdot p$ (see [2, Theorem 2.2]).

Theorem 2.1. *Let A be a C^*-algebra and let B be a C^*-subalgebra of A. Then the following conditions are equivalent*:

(1) *B is a hereditary C^*-subalgebra of A that generates an essential ideal of A.*

(2) *There exists a projection of norm one Q from A^{**} onto B^{**} such that Q is injective on $C(A_+)$ and $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on B.*

Proof. Let p be the identity of the von Neumann subalgebra B^{**} of A^{**}. Denote by $c(p)$ the central cover of p in A^{**}. Since p is a projection in A^{**}, $c(p)$ is also a projection in A^{**} (see [4, 2.6.2]). Since $A^{**}c(p) \cap A \supset pA^{**}p \cap A \supset B$, $A^{**}c(p) \cap A$ is an ideal of A including B. Let q be the open central projection of A^{**} corresponding to $I(B)$. Since q is a central projection in A^{**} majorizing p and $c(p)$ is the central cover of p, we see that $q \geq c(p) \geq p$. We thus see that $I(B) = A^{**}q \cap A \supset A^{**}c(p) \cap A \supset B$. Since it is clear that $I(B) \subset A^{**}c(p) \cap A$, we obtain that

$$I(B) = A^{**}q \cap A = A^{**}c(p) \cap A.$$

Since $A^{**}q \ni c(p)$ and since $A^{**}q \cap A$ is strongly dense in $A^{**}q$, it follows from the above equality that $c(p)$ belongs to the strong closure of $A^{**}c(p) \cap A$.

This means that $c(p)$ is open in A^{**} (see [4, 3.11.10]). Hence we conclude that $q = c(p)$. Thus the projection of norm one P from A^{**} onto $I(B)^{**}$ satisfying that $\overline{\varphi} \circ P = \overline{\varphi}$ for every state φ on $I(B)$ is given by the formula

$$P(\cdot) = c(p) \cdot c(p)$$

(see [2, Theorem 2.2]). Here note that $I(B)$ is essential in A if and only if the projection P is injective on A (see [3, Theorem 2.1]).

Now we show that the implication $(1) \Rightarrow (2)$. Let Q be a projection of norm one from A^{**} onto B^{**} such that $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on B. We have to show that Q is injective on $C(A_+)$. Assume that $Q(c(x)) = 0$ for a positive element x in A. We then obtain that $c(x)p = pc(x)p = Q(c(x)) = 0$. Since $c(c(x)p) = c(x)c(p) = c(xc(p))$ by [4, 2.6.4], we see that $P(x) = c(p)xc(p) = xc(p) = 0$. Since P is injective on A, we obtain that $x = 0$, i.e., $c(x) = 0$.

$(2) \Rightarrow (1)$. Let P be the projection from A^{**} onto $I(B)^{**}$ mentioned above. In order to obtain condition (1), we have only to show that P is injective on A. For $x \in A$, assume that $P(x) = c(p)xc(p) = 0$. We then have $x^*xc(p) = x^*c(p)xc(p) = 0$. Hence we have $c(x^*x)c(p) = c(x^*xc(p)) = 0$, from which follows that $c(x^*x)p = c(x^*x)c(p)p = 0$. We thus obtain that $Q(c(x^*x)) = pc(x^*x)p = 0$, which means that $c(x^*x) = 0$, i.e., $x^*x = 0$. Q.E.D.

For a C^*-algebra A, we denote by Z the center of the enveloping von Neumann algebra A^{**} of A.

Theorem 2.2. *Let A be a C^*-algebra and let B be a C^*-subalgebra of A. Then the following conditions are equivalent*:

(1) *B is a full hereditary C^*-subalgebra of A.*

(2) *There exists a projection of norm one Q from A^{**} onto B^{**} such that Q is injective on Z and $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on B.*

Proof. We can easily check that

$$\text{fullness of } B \Leftrightarrow I(B) = A \Leftrightarrow c(p) = 1.$$

$(1) \Rightarrow (2)$. Let Q be a projection of norm one from A^{**} onto B^{**} such that $\overline{\varphi} \circ Q = \overline{\varphi}$ for every state φ on B. For $x \in Z$, with $x \geq 0$, assume that $Q(x) = pxp = 0$. We then obtain that $xp = 0$. Using $c(p) = 1$ and [4, 2.6.4], we have $x = xc(p) = c(xp) = 0$. Thus Q is injective on the set of positive elements in Z.

For $x \in Z$, assume that $Q(x) = 0$. In order to prove that Q is injective on Z, we have only to show that $x^*x = 0$. Since we easily see that Q is multiplicative on Z, we have $Q(x^*x) = Q(x^*)Q(x) = 0$. Since x^*x is a positive element in Z, it follows from the above observation that $x^*x = 0$.

$(2) \Rightarrow (1)$. Since $1 - c(p) \in Z$, we have

$$Q(1 - c(p)) = p(1 - c(p))p = p - pc(p)p = 0.$$

By assumption, we obtain that $c(p) = 1$. Q.E.D.

This section ends by stating a remark concerning the above theorem.

Remark 2.3. Let A be a C^*-algebra and let B be a full hereditary C^*-subalgebra of A. The restriction of Q in condition (2) in Theorem 2.2 to the center of A^{**} is a homomorphism into the center of B^{**} (cf. the proof of the implication

(1) $\Rightarrow$ (2)). We now assert that the centers of A^{**} and B^{**} are isomorphic under Q. In fact, we see from Theorem 2.2 that Q is an isomorphism from the center of A^{**} *into* the center of B^{**}. Since B^{**} is the reduced von Neumann algebra of A^{**} with respect to p, the center of B^{**} is just the image of the center of A^{**} by Q.

3. C^*-CROSSED PRODUCTS OF HEREDITARY C^*-SUBALGEBRAS

By a C^*-dynamical system, we mean a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G and a group homomorphism α from G into the automorphism group of A such that $G \ni t \to \alpha_t(x)$ is continuous for each x in A. We denote by $A \times_\alpha G$ the *C^*-crossed product* of A by G, which is the enveloping C^*-algebra of the Banach *-algebra $L^1(A, G)$ of all Bochner integrable A-valued functions on G (see [4, 7.6]).

Let π be a representation of A on a Hilbert space H. Then the covariant representation $(\overline{\pi}, \lambda, L^2(H, G))$ is given by

$$(\overline{\pi}(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))\xi(t), \qquad (\lambda_s\xi)(t) = \xi(s^{-1}t)$$

for $a \in A$, $s \in G$, and $\xi \in L^2(H, G)$ where $L^2(H, G)$ denotes the Hilbert space of all square integrable H-valued functions on G. The *regular representation* of $A \times_\alpha G$ induced by (π, H) is the representation $(\overline{\pi} \times \lambda, L^2(H, G))$ defined by

$$((\overline{\pi} \times \lambda)(x)\xi)(t) = \int_G (\overline{\pi}(x(s))\lambda_s\xi)(t)\, ds$$

for $x \in L^1(A, G)$ and $\xi \in L^2(H, G)$ (see [4, 7.7]).

Let π be faithful. Then $(\overline{\pi} \times \lambda)(A \times_\alpha G)$ is called the *reduced C^*-crossed product* of A by G, which is denoted by $A \times_{\alpha r} G$ (cf. [4, 7.7]). Note that if G is amenable, then $A \times_{\alpha r} G$ is isomorphic to $A \times_\alpha G$ (see [4, 7.7.7]).

For every α-invariant C^*-subalgebra B of A, $B \times_{\alpha r} G$ is always a C^*-subalgebra of $A \times_{\alpha r} G$. But in general $B \times_\alpha G$ is not necessarily a C^*-subalgebra of $A \times_\alpha G$. If B is a hereditary C^*-subalgebra of A or if G is amenable, then $B \times_\alpha G$ is a C^*-subalgebra of $A \times_\alpha G$ (see [1, Lemma 3]).

Lemma 3.1. *Let (A, G, α) be a C^*-dynamical system. Let B be an α-invariant C^*-subalgebra of A. Suppose that $B \times_\alpha G$ is a C^*-subalgebra of $A \times_\alpha G$. Then $I(B) \times_\alpha G = I(B \times_\alpha G)$. Furthermore we have $I(B) \times_{\alpha r} G = I(B \times_{\alpha r} G)$.*

Proof. First remark that $I(B)$ (resp. $I(B \times_\alpha G)$) is the closed linear span of ABA (resp. $(A \times_\alpha G)(B \times_\alpha G)(A \times_\alpha G)$). Clearly we have

$$L^1(A, G)L^1(B, G)L^1(A, G) \subset L^1(I(B), G) \subset I(B) \times_\alpha G \subset A \times_\alpha G,$$

hence

$$(A \times_\alpha G)(B \times_\alpha G)(A \times_\alpha G) \subset I(B) \times_\alpha G.$$

Passing to the closed linear span of $(A \times_\alpha G)(B \times_\alpha G)(A \times_\alpha G)$, we obtain that $I(B \times_\alpha G) \subset I(B) \times_\alpha G$.

Now assume that $I(B) \times_\alpha G \neq I(B \times_\alpha G)$. Since $I(B \times_\alpha G)$ is an ideal, there exists a state φ of $A \times_\alpha G$ such that $\varphi \not\equiv 0$ on $I(B) \times_\alpha G$ and $\varphi \equiv 0$ on $I(B \times_\alpha G)$. Denote by $(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS representation of $A \times_\alpha G$ associated with φ (see [4, 3.3.3]). Let (π, u, H_φ) be the covariant representation of A corresponding to (π_φ, H_φ) (cf. [4, 7.6.4]). For $x, y \in A \times_\alpha G$ and $z \in B \times_\alpha G$, taking into account $y^* z x \in I(B \times_\alpha G)$, we have

$$(\pi_\varphi(z)\pi_\varphi(x)\xi_\varphi \mid \pi_\varphi(y)\xi_\varphi) = \varphi(y^* z x) = 0.$$

We thus conclude that $\pi_\varphi \equiv 0$ on $B \times_\alpha G$, which yields that $\pi \equiv 0$ on B. Hence we obtain that $\pi \equiv 0$ on $I(B)$. We now see that $\pi_\varphi = \pi \times u \equiv 0$ on $I(B) \times_\alpha G$, which means that $\varphi \equiv 0$ on $I(B) \times_\alpha G$. Thus we have reached a contradiction.

Next we show that $I(B) \times_{\alpha r} G = I(B \times_{\alpha r} G)$. Let π be a faithful representation of A on a Hilbert space H. Then the reduced crossed products $A \times_{\alpha r} G$ and $B \times_{\alpha r} G$ are just $(\overline{\pi} \times \lambda)(A \times_\alpha G)$ and $(\overline{\pi} \times \lambda)(B \times_\alpha G)$, respectively. Since

$$I(B) \times_{\alpha r} G = (\overline{\pi} \times \lambda)(I(B) \times_\alpha G) = (\overline{\pi} \times \lambda)(I(B \times_\alpha G)),$$

we have only to show that

$$(\overline{\pi} \times \lambda)(I(B \times_\alpha G)) = I(B \times_{\alpha r} G).$$

Here note that $(\overline{\pi} \times \lambda)(I(B \times_\alpha G))$ is an ideal of $A \times_{\alpha r} G$ containing $B \times_{\alpha r} G$ ($= (\overline{\pi} \times \lambda)(B \times_\alpha G)$). Since $I(B \times_{\alpha r} G)$ is the ideal generated by $B \times_{\alpha r} G$, we see that

$$(\overline{\pi} \times \lambda)(I(B \times_\alpha G)) \supset I(B \times_{\alpha r} G).$$

On the other hand, we have

$$(\overline{\pi} \times \lambda)((A \times_\alpha G)(B \times_\alpha G)(A \times_\alpha G)) = (A \times_{\alpha r} G)(B \times_{\alpha r} G)(A \times_{\alpha r} G) \subset I(B \times_{\alpha r} G).$$

Since $I(B \times_\alpha G)$ is spanned by $(A \times_\alpha G)(B \times_\alpha G)(A \times_\alpha G)$, we see that

$$(\overline{\pi} \times \lambda)(I(B \times_\alpha G)) \subset I(B \times_{\alpha r} G).$$

Thus we obtain the desired result. Q.E.D.

We now mention the hereditary C^*-subalgebra-version of the above result. Since we do not use such a result in the sequel and since the proof proceeds along lines similar to those of the proof of the preceding lemma (use [2, Theorem 2.2] to find a state playing the same role as φ in the above proof), we shall omit the proof and leave it to the reader.

Let A be a C^*-algebra and let B be a C^*-subalgebra of A. We denote by $H(B)$ the hereditary C^*-subalgebra of A generated by B. Then we have

Proposition 3.2. *Let (A, G, α) be a C^*-dynamical system. Let B be an α-invariant C^*-subalgebra of A. Suppose that $B \times_\alpha G$ is a C^*-subalgebra of $A \times_\alpha G$. Then $H(B) \times_\alpha G = H(B \times_\alpha G)$. Furthermore we have $H(B) \times_{\alpha r} G = H(B \times_{\alpha r} G)$.*

In [3, Proposition 2.4], it is shown that if I is an α-invariant C^*-subalgebra of A and if $I \times_\alpha G$ is an essential ideal of $A \times_\alpha G$, then I is an essential ideal of A. The next proposition is the reduced C^*-crossed product version for such a result.

Proposition 3.3. *Let (A, G, α) be a C^*-dynamical system. Let I be an α-invariant C^*-subalgebra of A. Consider the following conditions:*

(1) *I is an essential ideal of A.*

(2) *$I \times_\alpha G$ is an essential ideal of $A \times_\alpha G$.*

(3) *$I \times_{\alpha r} G$ is an essential ideal of $A \times_{\alpha r} G$.*

Then it follows that $(2) \Rightarrow (1) \Leftrightarrow (3)$. *In particular, if* G *is amenable, conditions* (1)–(3) *are equivalent.*

Proof. $(1) \Rightarrow (3)$. The second part of the proof of [3, Proposition 2.4] did not use amenability of G except to derive that the regular representations of the (enveloping) C^*-crossed product induced by faithful representations of the original C^*-algebra are also faithful. By the way, the regular representations of the reduced C^*-crossed product induced by faithful representations are always faithful (cf. [4, 7.7.5]). Hence the second part of the proof is valid, with $A \times_\alpha G$ and $I \times_\alpha G$ replaced by $A \times_{\alpha r} G$ and $I \times_{\alpha r} G$ respectively, for the proof of the implication $(1) \Rightarrow (3)$.

$(3) \Rightarrow (1)$. It follows from the proof of $(3) \Rightarrow (1)$ in [3, Corollary 2.3] that I is an ideal of A. Let J be a proper ideal of A such that $I \cap J = \{0\}$. In order to prove that I is essential in A, we have only to show that $J = \{0\}$. Let $\widetilde{J}$ be the α-invariant ideal of A generated by J. Since $I \cap \{\bigcup_{t\in G} \alpha_t(J)\} = \{0\}$, we see that $I \cap \widetilde{J} = \{0\}$. Then $\widetilde{J} \times_{\alpha r} G$ is an ideal of $A \times_{\alpha r} G$ and $(I \times_{\alpha r} G) \cap (\widetilde{J} \times_{\alpha r} G) = \{0\}$. This implies that $\widetilde{J} \times_{\alpha r} G = \{0\}$, since $I \times_{\alpha r} G$ is an essential ideal of $A \times_{\alpha r} G$. Hence we conclude that $\widetilde{J} = \{0\}$. Q.E.D.

In the above proof of $(3) \Rightarrow (1)$, there is no reason, a priori, why we have to restrict our attention to reduced C^*-crossed products. In fact, such a proof is valid for the C^*-crossed product case, hence we obtain an alternative proof of the implication $(2) \Rightarrow (1)$ already shown in [3, 2.4].

Let B be an α-invariant C^*-subalgebra of A. Then the following conditions (1)–(3) are equivalent:

(1) *B is a hereditary C^*-subalgebra of A.*

(2) *$B \times_\alpha G$ is a hereditary C^*-subalgebra of $A \times_\alpha G$.*

(3) *$B \times_{\alpha r} G$ is a hereditary C^*-subalgebra of $A \times_{\alpha r} G$.*

In fact, the implication $(1) \Rightarrow (2)$ was shown in [1, Theorem 4]. Since every reduced C^*-crossed product is a quotient of the C^*-crossed product, the implication $(2) \Rightarrow (3)$ follows from [4, 1.5.11]. Although the implication $(2) \Rightarrow (1)$ was given in [2, Lemma 3.1], the proof is exactly valid also for the implication $(3) \Rightarrow (1)$. Actually, we may modify the proof of [2, Lemma 3.1] as follows. Let a be an element in A with $b^*b \geq a^*a$ for some b^*b in B. Now assume that $a^*a \notin B$. There then exists a nonzero linear functional ψ on A such that $\psi(a^*a) \neq 0$ and $\psi\,|_B = 0$. Thus for a suitable faithful representation (π, H) of A, there exist vectors ξ_0 and η_0 in H such that $(\pi(a^*a)\xi_0 \mid \eta_0) \neq 0$ and $(\pi(c)\xi_0 \mid \eta_0) = 0$ for all c in B. Now we have $u_f^* b^* b u_f \geq u_f^* a^* a u_f$ in $A \times_\alpha G$ for all $f \in L^1(G)$. Since $B \times_{\alpha r} G$ is a hereditary C^*-subalgebra of $A \times_{\alpha r} G$ and since $(\overline{\pi} \times \lambda)(u_f^* b^* b u_f) \in B \times_{\alpha r} G$, we conclude that $(\overline{\pi} \times \lambda)(u_f^* a^* a u_f) \in B \times_{\alpha r} G$. But the proof of [2, Lemma 3.1] shows that $((\overline{\pi} \times \lambda)(u_f^* a^* a u_f)\xi \mid \eta) \neq 0$ for a suitable function f in $L^1(G)$ and some $\xi, \eta \in L^2(H, G)$, which contradicts that $((\overline{\pi} \times \lambda)(x)\xi \mid \eta) = 0$ for all $x \in B \times_{\alpha r} G$ and $(\overline{\pi} \times \lambda)(u_f^* a^* a u_f) \in B \times_{\alpha r} G$.

We are now in a position to present our theorems in this section.

Theorem 3.4. *Let* (A, G, α) *be a* C^**-dynamical system. Let* B *be an* α*-invariant* C^**-subalgebra of* A*. Consider the following conditions*:

(1) *B is a hereditary C^*-subalgebra of A that generates an essential ideal of A.*
(2) *$B \times_\alpha G$ is a hereditary C^*-subalgebra of $A \times_\alpha G$ that generates an essential ideal of $A \times_\alpha G$.*
(3) *$B \times_{\alpha r} G$ is a hereditary C^*-subalgebra of $A \times_{\alpha r} G$ that generates an essential ideal of $A \times_{\alpha r} G$.*

Then it follows that $(2) \Rightarrow (1) \Leftrightarrow (3)$. *In particular, if G is amenable, conditions* (1)–(3) *are equivalent.*

Proof. By Lemma 3.1, condition (2) (resp. (3)) is equivalent to the condition that $I(B) \times_\alpha G$ (resp. $I(B) \times_{\alpha r} G$) be essential in $A \times_\alpha G$ (resp. $A \times_{\alpha r} G$).

$(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$. It follows from Proposition 3.3 that $I(B)$ is essential in A. This implies condition (1).

$(1) \Rightarrow (3)$. Condition (1) is nothing but the condition that $I(B)$ be essential in A. Hence condition (3) follows again from Proposition 3.3. Q.E.D.

Theorem 3.5. *Let (A, G, α) be a C^*-dynamical system. Let B be an α-invariant C^*-subalgebra of A. Then the following conditions are equivalent*:
(1) *B is a full hereditary C^*-subalgebra of A.*
(2) *$B \times_\alpha G$ is a full hereditary C^*-subalgebra of $A \times_\alpha G$.*
(3) *$B \times_{\alpha r} G$ is a full hereditary C^*-subalgebra of $A \times_{\alpha r} G$.*

Proof. $(1) \Rightarrow (2)$ and (3). Since B is full, we have $I(B) = A$. Hence it follows from Lemma 3.1 that

$$I(B \times_\alpha G) = I(B) \times_\alpha G = A \times_\alpha G, \qquad I(B \times_{\alpha r} G) = I(B) \times_{\alpha r} G = A \times_{\alpha r} G.$$

This means conditions (2) and (3).

$(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$. Since $B \times_\alpha G$ (resp. $B \times_{\alpha r} G$) is full, we have $I(B \times_\alpha G) = A \times_\alpha G$ (resp. $I(B \times_{\alpha r} G) = A \times_{\alpha r} G$). Hence it follows from Lemma 3.1 again that $I(B) \times_\alpha G = A \times_\alpha G$ (resp. $I(B) \times_{\alpha r} G = A \times_{\alpha r} G$). An easy consequence of [4, 7.7.9] yields that $I(B) = A$, which means condition (1). Q.E.D.

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