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A NOTE ON THE CONE MULTIPLIER

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ABSTRACT. In this paper we study the convolution operator given on the Fourier transform side by multiplication by

$$m_\alpha(x, z) = \phi(z)(1 - |x|/z)_+^\alpha, \qquad (x, z) \in \mathbf{R}^2 \times \mathbf{R},\ \alpha > 0,$$

where $\phi \in C_0^\infty(1, 2)$. We will prove that m_α defines a bounded operator on $L^4(\mathbf{R}^3)$ if $\alpha > \frac{1}{8}$. Furthermore, as a generalization of a result of C. Fefferman (Acta Math. **124** (1970), 9–36), we will show that an (L^2, L^p) restriction theorem for compact C^∞ submanifolds $M \subset \mathbf{R}^n$ of arbitrary codimension imply results for multipliers having a singularity of the form $\operatorname{dist}(x, M)^\alpha$ near M.

INTRODUCTION

The purpose of this paper is to make some observations concerning the convolution operators $T_\alpha f = \check{m}_\alpha * f$ where the multiplier m_α has a singularity on a finite part of the light cone and is given by

$$m_\alpha(x, z) = \phi(z)(1 - |x|/z)_+^\alpha, \qquad (x, z) \in \mathbf{R}^2 \times \mathbf{R},\ \alpha > 0, \phi \in C_0^\infty(1, 2).$$

It is conjectured that for fixed α, the operator T_α is bounded on $L^p(\mathbf{R}^3)$ in the same p-range as the two-dimensional Riesz means of order α (see [17]). A well-known theorem of de Leeuw, which roughly speaking says that by restriction of a multiplier to any subspaces we obtain a multiplier for L^p corresponding to this subspace, gives us that T_α can be bounded only on the range where the Riesz means of order α in two dimensions are bounded. This is the case for $\alpha > \frac{1}{2}$ if $1 \le p \le \infty$ and as shown in [2, 11, 12] if $p = 4$ for $\alpha > 0$. By analytic interpolation of these two results one gets the full L^p boundedness range. By using asymptotic estimates for $\hat{m}_\alpha$ one can see that m_α gives a multiplier for L^1 if $\alpha > \frac{1}{2}$. Clearly only the behaviour of m_α near the finite part of the light cone, K, influences the L^p boundedness of the corresponding operator T_α. Therefore, we may think that our multiplier is given by the function $\phi(x)\operatorname{dist}(x, K)^\alpha$, where $\phi \in C_0^\infty$ is a suitable cut-off function. In general it is assumed that for multipliers of the form $\phi(x)\operatorname{dist}(x, M)^\alpha$, where M is a

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smooth compact hypersurface in $\mathbf{R}^n$ the L^p boundedness of the corresponding convolution operator depends on the vanishing behaviour of the principal curvatures on M. If M has one point x_0, where all principal curvatures do not vanish, then we expect that the corresponding multiplier gives a bounded operator on the same (α, p)-range as in case where M is a sphere in $\mathbf{R}^n$. In this case the range cannot be greater since the Fourier transform of the multiplier vanishes in a conical neighborhood of the normal at x_0 of the same order at infinity as the Fourier transform of the Riesz multiplier (see also [5]). If M is the surface of a cylinder, where one principal curvature always vanishes, then the corresponding operators essentially factor and its main part is a Riesz mean of order α in one dimension lower. If M is a hypersurface in $\mathbf{R}^3$ and one principal curvature always vanishes on M—the principle example here is the light cone—the situation is more subtle than in the case of the cylinder and Theorem (1.0) gives a first answer in this case. Lastly if M is a piece of a hyperplane, i.e., all principal curvatures vanish, we have L^p boundedness for $\alpha > 0$, $1 < p < \infty$. Thus one could expect that the more the principal curvatures vanish the greater the L^p boundedness range for a given α. This in some sense will be established by Theorem 1.0, but this may not be the sharpest result.

Further we will show a result for multipliers with a singularity of the form $\operatorname{dist}(x, M)^\alpha$, where M is a compact C^∞ submanifold of $\mathbf{R}^n$ of codimension $l \geq 1$, by assuming that an (L^2, L^p) restriction inequality holds for the Fourier transform with respect to M. The idea behind the proof is that we may use the restriction inequality for M and for translates of M in normal directions at points of M such that these translates of M fill out an open set. This method is sharp in a sense that there are examples for every codimension in which we get the optimal boundedness range (see [13]). For the cone multiplier m_α in $\mathbf{R}^3$ Theorem (2.0) gives us a result which is worse than (1.0). We mention further that Theorem 2.0 gives us a result for hypersurfaces that are of finite type at each point (see also [15] where an analog for Riesz means on compact manifolds is proved), but as is known the (L^2, L^p) restriction inequality holds in that case in a smaller range than, for example, in case of a sphere. The case of infinite vanishing on subsets of M in higher dimensions is unsolved. Only in two dimensions is there a complete answer (see [14]).

With respect to the notation we shall use the convention that C is a constant that is not the same at each occurrence

Main result

For the finite part of the cone multiplier we will show

Theorem 1.0. *If $\alpha > \frac{1}{8}$ then m_α gives a bounded operator on $L^4(\mathbf{R}^3)$.*

For the proof we remark that it is sufficient to show that for a smooth bump function ϕ_k with support contained in

$$\left\{x = (re^{i\theta}, z) \in \mathbf{R}^3 \middle| 1 \leq \frac{r}{z} \leq 1 + \frac{1}{2^k},\ 1 \leq z \leq 2,\ -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\right\}$$

we have

$$\|\check{\phi}_k * f\|_4 \leq C 2^{k/8} k^\gamma \|f\|_4,$$

where $\gamma > 0$ is some positive number (for a similar reduction step see [7, 11]). To show this inequality we make a further decomposition of the operator on the Fourier transform side in the angular variable θ. Let $N = 2^{k/2}$,

$$S_m = \left\{ (x, z) = (re^{i\theta}, z) \in \mathbf{R}^3 | \theta \in I_m = \left[\frac{m}{N}, \frac{m+1}{N} \right] \right\}, \qquad m = 0, \pm 1, \dots, \pm N,$$

$E_m = S_m \cap \operatorname{supp} \phi_k$, and choose smooth bump functions ψ_m essentially supported on E_m such that $\sum_m \psi_m = \phi_k$ and $|\partial_\theta^\alpha \partial_\xi^\beta \partial_\zeta^\gamma \psi_m| \le C_{\alpha,\beta,\gamma} N^\alpha N^{2\beta}$, for all $\alpha, \beta, \gamma \in \mathbf{N}$, where ζ denotes a vector that lies in E_m and ξ is normal to the surface of the cone inside of E_m. The theorem will follow by establishing the following three inequalities:

$$(1.1) \qquad \left\| \sum_m \check{\psi}_m * f \right\|_4 \le C N^{1/4} \left\| \left(\sum_m |\check{\psi}_m * f|^2 \right)^{1/2} \right\|_4,$$

$$(1.2) \qquad \left\| \left(\sum_m |\check{\psi}_m * f_m|^2 \right)^{1/2} \right\|_4 \le C(\log N)^\gamma \left\| \left(\sum_m |f_m|^2 \right)^{1/2} \right\|_4,$$

$$(1.3) \qquad \left\| \left(\sum_m |f_m|^2 \right)^{1/2} \right\|_4 \le C(\log N)^{\gamma'} \|f\|_4,$$

where in (1.3) $f_m = \check{\chi}_{S_m} * f$ and γ, γ' are certain positive constants. Inequality (1.3) is proved in [8]. For inequality (1.2) we remark that, by the uncertainty principle, $\check{\psi}_m * f(x)$ is essentially dominated by the mean value $M_m f(x)$ of f over a rectangle containing x and of dimensions $1 \times N \times N^2$ whose longest side lies in direction $(e^{im/N}, -1)$, i.e., normal to E_m. Therefore (1.2) follows from the bound

$$\left\| \sup_m M_m f \right\|_2 \le C(\log N)^{2\gamma} \|f\|_2$$

for which we refer to [9]. A slightly weaker inequality, where the $\log N$ term is replaced by $C_\varepsilon N^\varepsilon$, $\varepsilon > 0$, can be obtained by modifying a little bit of the argument in [4] for the Kakeya maximal function, i.e., instead of taking the supremum over a discrete set of angles one takes it over all angles, and after smoothing the rectangles we bound the supremum over the angles by an L^2 Sobolev norm, an application of Plancherel's theorem then gives what we want and this is sufficient for our purpose. It remains to show (1.1), which is an inequality estimating the cancellation on the left side. Using a duality argument and applying Plancherel's theorem we get for the square of the left side of (1.1)

$$\left|\int_{\mathbf{R}^3}\left(\sum \check{\psi}_m * f\right)^2 g\,dx\right| = \left|\int \sum_{m,m'} \psi_m\hat{f} * \psi_{m'}\hat{f}\check{g}\,dx\right|$$

$$\leq \int \sum |\check{\psi}_m * f|^2 \left(\sum_{m,m'} |(\chi_{E_m+E_{m'}}\check{g})^{\widehat{}}|^2\right)^{1/2} dx$$

$$\leq \left\|\left(\sum_m |\check{\psi}_m * f|^2\right)^{1/2}\right\|_4^2 \left\|\left(\sum_{m,m'} |(\chi_{E_m+E_{m'}}\check{g})^{\widehat{}}|^2\right)^{1/2}\right\|_2 .$$

We are done if the following inequality is established:

$$(*) \qquad \sum_{m,m'} \chi_{E_m+E_{m'}}(x) \leq CN \quad \text{for all } x \in \mathbf{R}^3 .$$

To prove it, we first translate the situation along the z-axis so that the sets E_m lie between the planes $z = \frac{1}{2}$ and $z = -\frac{1}{2}$. A simple symmetry argument shows now that it is sufficient to prove $(*)$ for $x = (r, 0, 0)$, $1 < r < 3$. It is also easy to see that if A, B lie in E_m, $E_{m'}$, respectively, then in order to get $A + B = x$ their z-coordinates must have different signs and the same magnitude independent of m, m'. Let $\mathscr{P}(t)$ be the set of pairs (m, m') such that there are points $A \in E_m$, $B \in E_{m'}$, with $A+B = x$, and the z-coordinate of A is t. It remains to show

$$\mathscr{S} = \#\left\{\bigcup_{t\in[-1/2,1/2]} \mathscr{P}(t)\right\} \leq CN .$$

The orthogonal projection onto the x, y-plane of $E_m \cap \{z = t\}$ is of course contained in the rectangle

$$R_m(t) = e^{im/N}\{1 + t + R\},$$

where $R = \{x + iy \mid |x| \leq 1/N^2,\ |y| \leq 1/N\}$. It follows that

$$R_m(t) + R_{m'}(-t) \subset 2e^{i(m+m')/2N}\left\{\left(\cos\frac{m-m'}{2N} + \frac{4(m'-m)}{N^2}[-1, 1]\right)\right.$$
$$\left.\times\left(t\sin\frac{m-m'}{2N} + \frac{4}{N}[-1, 1]\right)\right\} .$$

By setting $k = (m+m')/2$ and $l = (m-m')/2$ and approximating the trigonometric functions to second order we see that for $(r, 0)$ to lie in $R_m(t)+R_{m'}(-t)$ it is necessary that

(1) $N^2(r-2) + l^2 - rk^2 \in 8l[-1, 1]$,
(2) $rk - 2tl \in 8[-1, 1]$.

Condition (2) is not significant since if $t \in [-\frac{1}{2}, \frac{1}{2}]$ varies then the strips defined by (2) cover a big sector of $\mathbf{Z}^2 \cap [-N, N]^2$ determined by r. We are left with condition (1) which says that the pairs $(k, l) \in \mathbf{Z}^2 \cap [-N, N]^2$ we have to count lie at a distance at most 4 from the hyperbolic curve $l =$

$\sqrt{rk^2 + 16 + (2-r)N^2}$. Since the length of this curve in $[-N, N]^2$ is $\approx N$ we get $\mathscr{S} \le CN$. □

It is not hard to see that inequality $(*)$ is sharp, therefore, the method above cannot lead to an improvement of (1.1).

Let us remark that by simple arguments Theorem (1.0) is equivalent to the boundedness on $L^4(\mathbf{R}^3)$ of the convolution operator defined by the multiplier

$$M_\alpha(x, z) = e^{-z}(z - |x|)_+^\alpha$$

whose Fourier transform is given by

$$\widehat{M_\alpha}(x, z) = (1 + iz)^{-\alpha}[(1 + iz)^2 + |x|^2]^{-3/2}.$$

Obviously $\widehat{M_\alpha}$ has its singular behaviour near the light cone $z = |x|$ and we see that $\widehat{M_\alpha}$ has a quite subtle cancellation property. As remarked in the introduction we have $\hat{m}_\alpha \in L^1(\mathbf{R}^3)$ for $\alpha > \frac{1}{2}$. This is easily seen from the corresponding property of $\widehat{M_\alpha}$. We also remark that it can be shown by a rotation method that for $\alpha > \frac{1}{2}$

$$K_\alpha(x, z) = \left(1 - \frac{|x|}{z}\right)_+^\alpha, \qquad (x, z) \in \mathbf{R}^2 \times \mathbf{R},$$

gives a multiplier for $L^p(\mathbf{R}^3)$, $1 < p < \infty$. To see this let us first explain it for the Riesz multipliers. Let $\phi_\delta \in C^\infty(\mathbf{R}^2)$ be a bump function supported on a rectangle of size $\delta \times \sqrt{\delta}$ which lies tangential to the unit sphere S^1. Obviously

$$\int_{\mathrm{SO}(2)} \phi_\delta(gx)\, dg = \sqrt{\delta}\psi_\delta(|x|)$$

where ψ_δ is a radial bump function supported on a spherical shell of width δ. Of course, ϕ_δ is an $L^1(\mathbf{R}^2)$ multiplier with norm independent of δ hence ψ_δ is an $L^1(\mathbf{R}^2)$ multiplier with norm $\delta^{-1/2}$ and therefore an argument used in the proof of (1.0) gives that $(1 - |x|)_+^\alpha$ is a multiplier for $L^1(\mathbf{R}^2)$ if $\alpha > \frac{1}{2}$. The same method works for K_α. Here we consider at first for $g \in \mathrm{SO}(2)$ the homogeneous function $\phi_\delta(\frac{gx}{z})$, $(x, z) \in \mathbf{R}^2 \times \mathbf{R}$, and see by Marcinkievic's multiplier theorem that ϕ_δ gives a multiplier for $L^p(\mathbf{R}^3)$, $1 < p < \infty$, with norm independent of δ and we conclude as above that K_α gives a multiplier for $L^p(\mathbf{R}^3)$, $1 < p < \infty$, and $\alpha > 1/2$.

Let us now show that an (L^2, L^p) restriction inequality for a compact C^∞ submanifold M always implies a result for multipliers with a singularity of the form $\mathrm{dist}(x, M)^\alpha$ near M. The method is similar to that of [11, 16]. Assuming that our multipliers have only a singularity near M means that we cut the distance function with a suitable $\phi \in C_0^\infty$ so that our multiplier is given by $m_\alpha(x) = \phi(x)\mathrm{dist}(x, M)^\alpha$, where the support of ϕ is sufficiently small so that m_α is C^∞ away from M, i.e., in case of M being the unit sphere we avoid a discussion of the origin.

Theorem 2.0. *Let M be a submanifold of $\mathbf{R}^n$ of codimension l, and assume the restriction inequality for the Fourier transform*:

$$\|\hat{f}\|_{L^2(M)} \le C\|f\|_{L^p(\mathbf{R}^n)}$$

holds for $1 \le p \le p_0$. *Then* m_α *is a multiplier for* $L^p(\mathbf{R}^n)$ *if* $\alpha > n|1/2 - 1/p| - l/2$ *and* $1 \le p \le p_0$.

Remarks. For submanifolds of codimension l it follows by generalizing A. Knapp's homogeneity argument that the above restriction inequality can only hold for $p_0 \le 2(n+l)/(n+3l)$.

In [13] we proved an optimal (L^2, L^p) restriction inequality for certain homogeneous submanifolds using sharp asymptotic estimates for generalized Bessel functions given in [6] and using the asymptotic expansion in [1] we showed that Theorem 2.0 is sharp in these cases. This is done by checking the asymptotic behaviour of $\hat{m}_\alpha$ in a conical set and by using the fact that for a compactly supported multiplier for L^p its Fourier transform has to be in L^p. The $\mathrm{SO}(2) \times \mathrm{SO}(n)$-orbit $M = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n \,|\, |x|^2 + |y|^2 = 1, |x|^2|y|^2 - (x \cdot y)^2 = 2\}$ is such an example in case of 2 codimension.

For certain exponents α it could be that our multiplier has no singularity, e.g., if $\alpha = 2$. In that case one should introduce complex powers to maintain the singular behaviour of m_α near M.

For the proof of Theorem 2.0 we consider at first the multiplier m_α near a suitable small neighborhood W of a point $x_0 \in M$, such that on W we have a nice parameterization. Since M is compact, we can reduce the things to a local consideration. So we cut m_α with a suitable C_0^∞ function supported on W. We may also assume that $x_0 = 0$ and the tangent plane $T_{x_0}M$ is given by $\{(x_1, \dots, x_n) \in \mathbf{R}^n | x_{n-l+1} = \cdots = x_n = 0\} \cong \mathbf{R}^{n-l}$. Let

$$\gamma\colon \mathbf{R}^{n-l} \supset U \ni y \mapsto (y, \gamma(y)) \in M,$$

where $\gamma \in C^\infty(U, \mathbf{R}^l)$, $\gamma(0) = 0$, U open, a parameterization of M near x_0. Then

$$\Gamma\colon (y, v) \mapsto (y, v + \gamma(y))$$

defines a local diffeomorphism of an open set $U \times V \subset \mathbf{R}^{n-l} \times \mathbf{R}^l$ containing the origin of $\mathbf{R}^n$ onto $\Gamma(U \times V)$ that contains W as we may assume. This simply follows from the fact that $\det \Gamma' = 1$. Let ϕ be a C_0^∞ function on $\mathbf{R}^l$ such that

$$\phi(v) = \begin{cases} 1 & \text{for } 0 \le |v| \le 1, \\ 0 & \text{for } |v| > 2 \end{cases}$$

and let $\phi_k(v) = \phi(2^k v) - \phi(2^{k+1} v)$. Of course we have $\operatorname{supp} \phi_k = \{v \in \mathbf{R}^l | 1/2^{k+1} \le |v| \le 1/2^{k-1}\}$ and $\sum_{k \ge 0} \phi_k(v) = 1$ for $|v| \le 1$. Let Ψ be a function in $C_c^\infty(W)$ that cuts m_α as mentioned above, then $\operatorname{supp} \Psi \subset \Gamma(U \times V)$, and we may assume that $V \subset \{|v| \le 1\} \subset \mathbf{R}^l$. We set for $y \in U$ and $v \in V$:

$$\psi_k(x = \Gamma(y, v)) = \Psi(\Gamma(y, v))\phi_k(v).$$

Then ψ_k is supported in $B_k = \{v + M \subset \mathbf{R}^n | 1/2^{k+1} \le |v| \le 1/2^{k-1}\} \cap \operatorname{supp} \Psi$ and we have

$$m_\alpha(x) = \sum \psi_k(x) m_\alpha(x) = \sum m_k$$

where $|\partial^\beta m_k| \le C 2^{k|\beta|}$, using $|\partial^\beta \operatorname{dist}(x, M)^a| \le c \operatorname{dist}(x, M)^{-|\beta|}$ on B_k. This inequality for the derivatives of m_k implies that the Fourier transform of m_k is essentially supported on a ball of diameter $2^{k(1+\varepsilon)}$, $\varepsilon > 0$. Modulo negligible operators it is then sufficient to show that for a cube $Q \subset \mathbf{R}^n$ of side length $4 \cdot 2^{k(1+\varepsilon)}$ the operator $T_k f = \chi_Q \check{m}_k * f$ has the norm $2^{(n|1/2-1/p|-l/2)} C_\varepsilon 2^{\varepsilon k}$,

$\varepsilon > 0$, on $L^p(Q^*)$, here Q^* denotes the doubling of Q. By using a lemma of M. Christ [3, p. 18] the matter is reduced to the inequality

$$\|T_k f\|_{p_0'} \le C 2^{-(\alpha+l/2)k} \|f\|_2 .$$

To prove this inequality we first compute, setting $\overline{v} = (0, v) \in \mathbf{R}^{n-l} \times \mathbf{R}^l$,

$$\begin{aligned} |T_k f(t)| &= \left| \int m_k(x) \hat{f}(x) e^{-it\cdot x}\, dx \right| \\ &= \left| \int_{B_k} \int_U m_k(\Gamma(y, v)) \hat{f}(\Gamma(y, v)) e^{-it\cdot\Gamma(y, v)}\, dy\, dv \right| \\ &= \left| \int_{B_k} e^{-it\cdot\overline{v}} \int_U m_k \circ \Gamma \hat{f} \circ \Gamma e^{-it\cdot\Gamma(y, 0)}\, dy\, dv \right| \\ &\le \int_{B_k} \left| \int_U (m_k \hat{f}) \circ \Gamma e^{-it\cdot\Gamma(y, 0)}\, dy \right| dv . \end{aligned}$$

Now the restriction inequality for the Fourier transform with respect to M comes in. By dualizing it we get

$$\left\| \int_U (m_k \hat{f}) \circ \Gamma e^{-it\cdot\Gamma(y, 0)}\, dy \right\|_{L^{p_0'}(\mathbf{R}^n)} \le C \|(m_k \hat{f}) \circ \Gamma\|_{L^2(U)} .$$

Noting that $B_k \subset \{x \in \mathbf{R}^n | c_1/2^{k+1} \le \operatorname{dist}(x, M) \le c_2/2^{k-1}\}$, where c_1, c_2 are suitable constants, we obtain

$$\begin{aligned} \|T_k f\|_{p_0'} &\le C \int_{B_k} \|m_k(\Gamma(\cdot, v)) \hat{f} \circ \Gamma(\cdot, v)\|_{L^2(U)}\, dv \\ &\le C |B_k|^{1/2} \left(\int_{B_k} \int_U |m_k(\Gamma(y, v))|^2 |\hat{f} \circ \Gamma(y, v)|^2\, dy\, dv \right)^{1/2} \\ &\le C |B_k|^{1/2} 2^{-\alpha k} \left(\int_{B_k} \int_U |\hat{f} \circ \Gamma(y, v)|^2\, dy\, dv \right)^{1/2} \\ &\le C 2^{-k(l/2+\alpha)} \left(\int_{B_k} \int_U |\hat{f} \circ \Gamma(y, v)|^2\, dy\, dv \right)^{1/2} \\ &\le C 2^{-k(l/2+\alpha)} \left(\int_{\Gamma(B_k \times U)} |\hat{f}|^2 \right)^{1/2} \\ &\le C 2^{-k(l/2+\alpha)} \|f\|_{L^2(\mathbf{R}^n)} . \quad \square \end{aligned}$$

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