MAJORIZATION AND DOMINATION IN THE BERGMAN SPACE

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Abstract. Let \( f \) and \( g \) be functions analytic on the unit disk and let \( \| \cdot \| \) denote the Bergman norm. Conditions are identified under which there exists an absolute constant \( c \), with \( 0 < c < 1 \), such that the relationship \( |g(z)| \leq |f(z)| \) \((c \leq |z| < 1)\) will imply \( \|g\| \leq \|f\| \).

1. Introduction

Let \( \mathbb{C} \) denote the complex plane, \( \mathbb{D} \) the open unit disk, and \( L^2(\mathbb{D}) \) the Hilbert space of all measurable functions \( f: \mathbb{D} \to \mathbb{C} \) with
\[
\|f\|^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f|^2 \, dm < \infty,
\]
where \( dm \) denotes the Lebesgue area measure. The Bergman space \( A_2 \) is defined to be the subspace of \( L^2(\mathbb{D}) \) consisting of functions analytic on \( \mathbb{D} \).

Let \( f \) and \( g \) be analytic on \( \mathbb{D} \). We say \( g \) is majorized by \( f \) on a region \( R \subseteq \mathbb{D} \) if \( |g(z)| \leq |f(z)| \) for all \( z \in R \). By the positivity of \( dm \), if \( g \) is majorized by \( f \) on \( \mathbb{D} \) then certainly
\[
\|g\| \leq \|f\|.
\]
What other cases of majorization will imply (1)? In particular, when does majorization on an annulus imply (1)? That is, we investigate the existence of an absolute constant \( c \), with \( 0 < c < 1 \), such that if
\[
|g(z)| \leq |f(z)| \quad (c \leq |z| < 1)
\]
then
\[
\|g\| \leq \|f\|.
\]

In the case that either function is a monomial \( z^n \), it has been shown (see [3]) that (2) implies (3) for any \( c \leq 1/\sqrt{3} \).

For arbitrary \( f \) and \( g \) analytic on \( \mathbb{D} \), let \( Z_f \) and \( Z_g \) denote the zero sets (counting multiplicity) of \( f \) and \( g \), respectively. If \( Z_f \setminus Z_g \) is empty, then (2) implies (3) for any \( c \in (0, 1) \), by the classical maximum principle.

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Furthermore, it is known (see [1]) that when \( Z_g \setminus Z_f \) is empty (2) will again imply (3) for \( c \leq 1/(2e^2) \).

We seek conditions under which majorization on an annulus of inner radius \( c \in (0, 1) \) will imply (3), without requiring either relative complementary zero set to be empty. We will show that such a \( c \) exists when \( Z_g \setminus Z_f \) and \( Z_f \setminus Z_g \) are separated by an annulus.

2. Preliminary results

First we present some definitions and preliminary facts that will be appealed to throughout this article (see [1, 2]). Let \( H^\infty \) be the space of all bounded analytic functions on \( \mathbb{D} \), with

\[
\|h\|_\infty = \sup\{|h(z)| \mid z \in \mathbb{D}\}.
\]

For \( G, F \in L^2(\mathbb{D}) \), we say that \( G \) is dominated by \( F \) if \( \|Gh\| \leq \|Fh\| \) for all \( h \in H^\infty \), and we write \( G \prec F \). It follows that if \( G \prec F \) and \( G, F \in L^\infty(\mathbb{D}) \), then \( \|Gh\| \leq \|Fh\| \) for all \( h \in A_2 \). The following properties are direct consequences of the definition of domination.

**Property 1.** If \( G, F \in L^2(\mathbb{D}) \) and \( G \prec F \), then \( G \circ \phi \prec F \circ \phi \) for all M"obius transformations \( \phi \) on \( \mathbb{D} \).

**Property 2.** If \( G_i, F_i \in H^\infty \) and \( G_i \prec F_i \) for \( i = 1, \ldots, n \), then

\[
(G_1 G_2, \ldots, G_n) \prec (F_1 F_2, \ldots, F_n).
\]

We denote \( B_a(z) = \frac{a(a-z)}{|a|^2(a-az)} \) for \( a \in \mathbb{D}, a \neq 0 \), and \( B_0(z) = -z \).

**Proposition 1.** Let \( a \in \mathbb{D} \) and \( \gamma > 0 \). Then

\[
|B_a|^{\gamma((1+|a|)/(1-|a|))} < |z|^{\gamma},
\]

and

\[
\exp\{-2\gamma(1+z)/(1-z)\} < |z|^{\gamma}.
\]

**Corollary 1.** Let \( a \in \mathbb{D} \) and define \( \alpha = (1-|a|)/(1+|a|) \), \( \beta = (1+|a|)/(1-|a|) \). Then \( |z|^{\beta} \prec B_a \prec |z|^\alpha \).

(Proposition 1 is proved in [2] while Corollary 1 follows from (4) and Property 1.)

**Corollary 2.** Let \( B(z) = \prod_{k=1}^N B_{a_k}(z), |a_k| < 1 \), and let

\[
\beta = \sum_{k=1}^N \left( \frac{1+|a_k|}{1-|a_k|} \right) + 1,
\]

where \( N < \infty \) and \([\cdot]\) denotes the greatest integer function. Then \( |z|^\beta \prec B \).

**Proof.** By Corollary 1, we have for each \( k \) that \( |z|^{\beta_k} \prec B_{a_k} \) where \( \beta_k = (1+|a_k|)/(1-|a_k|) \). Since \( \beta_k \leq ([\beta_k] + 1) \), we have

\[
|z|^{([\beta_k]+1)} \prec B_{a_k}.
\]

Since both sides of (6) are in \( H^\infty \), we can apply Property 2 for \( k = 1, \ldots, N \), which yields \( |z|^\beta \prec B \). This proves the corollary. \( \square \)
Proposition 2 (see [1]). Suppose $G \in H^\infty$, $\|G\|_\infty \leq 1$, and $G(z) \neq 0$ for all $z \in \mathbb{D}$. If $|G(0)| \leq e^{-2\gamma}$ for $\gamma > 0$, then $G \prec |z|^\gamma$.

Proof. We have

$$G(z) = \lambda \exp \left\{ - \int_{\partial \mathbb{D}} \frac{\zeta + z}{\bar{\zeta} - z} \, d\mu(\zeta) \right\},$$

where $d\mu$ is a positive Borel measure on $\partial \mathbb{D}$, with $\mu(\partial \mathbb{D}) = -\log |G(0)| \geq 2\gamma$, and $|\lambda| = 1$. Let $\eta = \mu(\partial \mathbb{D})$. Using the generalized arithmetic-geometric mean inequality combined with the Fubini Theorem and (5), we obtain for all $h \in A^2$

$$\int_\mathbb{D} |Gh|^2 \, dm = \int_\mathbb{D} \exp \left\{ -2\eta \int_0^{2\pi} \Re \frac{e^{it} + z}{e^{it} - z} \cdot \frac{d\mu(t)}{\eta} \right\} |h(z)|^2 \, dm \leq \int_\mathbb{D} |Gh|^2 \, dm.$$

3. Main results

Theorem 1. Let $G \in H^\infty$ be such that $\|G\|_\infty \leq 1$ and $G(z) \neq 0$ for all $|z| < c$ for some $c \in (0, 1)$. If $|G(0)| < (c(1+c)/(1-c))^{\gamma}$ for some $\gamma > 0$, then $G \prec |z|^\gamma$.

(It is interesting to note that Proposition 2 is the limiting case of Theorem 1 since $c(1+c)/(1-c) \to e^{-2}$ as $c \to 1^-$. We postpone the proof of Theorem 1 until the end of the article. Once this theorem is proved, we can obtain the following results.

Theorem 2. Let $B(z) = \prod_{k=1}^N B_{a_k}(z)$, where $|a_k| < c < 1$. Suppose $G \in H^\infty$, $\|G\|_\infty \leq 1$, and $G(z) \neq 0$ for all $|z| < d$, with $d > c$. If

$$|G(0)| < (d^{(1+d)/(1-d)})^{2N},$$

then $G \prec B$.

Proof. We have that $G \prec |z|^{2N}$, by Theorem 1. Since $|a_k| < 1/3$, it follows that

$$\beta = \sum_{k=1}^N \left( \frac{1 + |a_k|}{1 - |a_k|} + 1 \right) = 2N.$$

Thus, $|z|^{2N} \prec B$, by Corollary 2. Therefore, $G \prec |z|^{2N} \prec B$, which proves the theorem. □

Theorem 3. Let $f, g \in A_2$. There exists an absolute constant $c_0$, with $0 < c_0 < 1$, such that for any $c < c_0$, if

(i) $|g(z)| \leq |f(z)|$ ($c \leq |z| < 1$) and

(ii) $Z_g \setminus Z_f \subset \{ z \in \mathbb{D} | c^{1/3} < |z| < 1 \}$,

then $g \prec f$. 

Proof. Let $B$ be the finite Blaschke product

$$B = \prod_{k=1}^{N} B_{a_k}, \quad \text{where } \{a_k\}_{k=1}^{N} = Z_f \setminus Z_g.$$ 

Note that (i) implies $|a_k| < c$ for $k = 1, \ldots, N$. Consider the function $G = gB/f$. It follows that $G \in H^\infty$, $\|G\|_{\infty} \leq 1$, and $G(z) \neq 0$ for $|z| < c^{1/3}$. Also, on $|z| = c$, we have

$$|G(z)| \leq |B(z)| \leq (2c)^N.$$ 

The classical maximal principle implies that $|G(0)| \leq (2c)^N$. Letting $d = c^{1/3}$, it follows that $2c \leq d^{2(1+d)/(1-d)}$ for $c < 0.0021 = c_0$. Thus,

$$|G(0)| \leq \left(d^{(1+d)/(1-d)}\right)^2 N$$

and $G(z) \neq 0$ for all $|z| < d$. Applying Theorem 2, we have $G \prec B$ or $\|Gh\| \leq \|Bh\|$ for all $h \in A_2$. Taking $h = h_1 f/B$, with $h_1 \in H^\infty$, we have $g \prec f$.  

Corollary 3. Let $f$ and $g$ be analytic on $D$. There exists an absolute constant $c_0$, with $0 < c_0 < 1$, such that for any $c < c_0$, if

\begin{enumerate}
  \item[(i)] $|g(z)| \leq |f(z)|$ (c $< |z| < 1$) and
  \item[(ii)] $Z_g \setminus Z_f \subset \{z \in D | c^{1/3} < |z| < 1\}$,
\end{enumerate}

then $\|g\| \leq \|f\|$.

Proof. If $\|f\| = \infty$, then there is nothing to prove. Thus we can assume that $f \in A_2$, which implies $g \in A_2$ by (i). Applying Theorem 3, we have $g \prec f$ and, in particular, $\|g\| \leq \|f\|$.  

Notice that (i) and (ii) together imply that $Z_g \setminus Z_f$ is separated from $Z_f \setminus Z_g$ by the annulus $\{z \in D | c \leq |z| \leq c^{1/3}\}$.

Proof of Theorem 1. Let $F = G/B$ where $B = \prod_{k=1}^{M} B_{a_k}$ and $\{a_k\}_{k=1}^{M} = Z_G$ ($M \leq \infty$). Applying (4), we have for $\lambda, \gamma > 0$

$$|B_{a_k}|^{\frac{1}{\gamma_0}} \prec |z|^\gamma$$

where $\alpha_k = (1 + |a_k|)/(1 - |a_k|)$. Since $F$ has no zeros in $D$ and $\|F\|_{\infty} \leq 1$, we can define the analytic function $\tilde{F} = F^{2\gamma/\log(1/|F(0)|)}$ that satisfies $\|\tilde{F}\|_{\infty} \leq 1$ and $|\tilde{F}(0)| = e^{-2\gamma}$. Thus, we can apply Proposition 2 to $\tilde{F}$ yielding

$$\tilde{F} \prec |z|^\gamma.$$ 

Now let $h \in H^\infty$. We have

$$\int_D |Gh|^2 \, dm = \int_D \left| \prod_k B_{a_k}(z) \right|^2 |F(z)h(z)|^2 \, dm$$

$$= \int_D \prod_k (|B_{a_k}(z)|^{2\gamma_0})^{1/(\lambda \gamma_0)} (|\tilde{F}(z)|)^2 \left(\frac{\log 1/|F(0)|}{2\gamma}|h(z)|^2 \right) \, dm,$$

where we choose

$$\lambda = \frac{1}{\gamma} \left[ \sum_k \left( \frac{1}{\alpha_k} \right) + \frac{1}{2} \log \frac{1}{|F(0)|} \right].$$
For this $\lambda$, we can apply the arithmetic-geometric mean inequality to (10) to obtain

$$\int_D |Gh|^2 \, dm \leq \sum_k \frac{1}{\lambda \gamma \alpha_k} \int_D (|B_{\alpha_k}^{\gamma \lambda} | h(z)|)^2 \, dm$$

$$+ \frac{\log(1/|F(0)|)}{2\lambda \gamma} \int_D |\tilde{F}(z)h(z)|^2 \, dm.$$  

Applying (7) and (8), it follows that

$$\int_D |Gh|^2 \, dm \leq \left[ \sum_k \left( \frac{1}{\lambda \gamma \alpha_k} + \frac{\log(1/|F(0)|)}{2\lambda \gamma} \right) \right] \int_D |z|^{2\gamma}|h(z)|^2 \, dm$$

$$= \int_D |z|^{2\gamma}|h(z)|^2 \, dm,$$

by the above choice of $\lambda$. We will show that $\lambda \geq 1$. Once this is done, it follows that

$$\int_D |Gh|^2 \, dm \leq \int_D |z|^{2\gamma}|h(z)|^2 \, dm,$$

which is the desired result. To see that $\lambda \geq 1$, first observe that

$$|F(0)| \prod_k |a_k| = |F(0)B(0)| = |G(0)| \leq e^{\gamma(1+c)/(1-c)},$$

where the last inequality follows by hypothesis. This implies

$$\frac{1+c}{1-c} \log \frac{1}{c} \leq \frac{1}{\gamma} \sum_k \left( \log \frac{1}{|a_k|} \right) + \frac{1}{\gamma} \log \frac{1}{|F(0)|}.$$  

It can be shown that the function $\phi(r) = \frac{1-1/r}{1+c} \log \frac{1}{r}$ is decreasing and $\phi(r) > 2$ on $(0, 1)$. We have, by assumption, that $|a_k| > c$ and thus, $\phi(|a_k|) \leq \phi(c)$ for all $k$. This yields

$$\log \frac{1}{|a_k|} \leq \left( \frac{1-|a_k|}{1+|a_k|} \right) \frac{1+c}{1-c} \log \frac{1}{c} = \frac{1}{\alpha_k} \left( \frac{1+c}{1-c} \log \frac{1}{c} \right).$$

This, together with (12), implies

$$\frac{1+c}{1-c} \log \frac{1}{c} \leq \frac{1}{\gamma} \left[ \sum_k \left( \frac{1}{\alpha_k} \right) \frac{1+c}{1-c} \log \frac{1}{c} + \log \frac{1}{|F(0)|} \right]$$

or

$$1 \leq \frac{1}{\gamma} \left[ \sum_k \left( \frac{1}{\alpha_k} \right) + \frac{2(1-c)}{(1+c) \log 1/c} \cdot \frac{1}{2} \log \frac{1}{|F(0)|} \right]$$

$$\leq \frac{1}{\gamma} \left[ \sum_k \left( \frac{1}{\alpha_k} \right) + \frac{1}{2} \log \frac{1}{|F(0)|} \right] = \lambda,$$

where the last inequality is a consequence of $\phi(c) > 2$. This completes the proof of the theorem. □
References


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