A GENERALIZATION OF THE PUNCTURED NEIGHBORHOOD THEOREM

WOO YOUNG LEE

(Communicated by Palle E. T. Jorgensen)

Abstract. If $T \in \mathcal{L}(X)$ is regular on a Banach space $X$, with finite-dimensional intersection $T^{-1}(0) \cap T(X)$, and if $S, S'$ are invertible, commute with $T$ and have sufficiently small norm, then $\dim(T - S')^{-1}(0) = \dim(T - S)^{-1}(0)$ and $\dim X/(T - S')X = \dim X/(T - S)X$.

In [5], Lee proved that if $T$ is a regular operator with some finite-dimensional intersection property on a Banach space and if 0 is the boundary of the spectrum of $T$, then 0 is an isolated point of the spectrum of $T$.

In this note we derive a generalization of the punctured neighborhood theorem and then strengthen the above result.

Throughout this note suppose $X$ and $Y$ are complex Banach spaces, write $\mathcal{L}(X, Y)$ for the set of bounded linear operators from $X$ to $Y$, and abbreviate $\mathcal{L}(X, X)$ to $\mathcal{L}(X)$. If $T \in \mathcal{L}(X)$ then we write $\sigma(T)$ for the spectrum of $T$. If $K$ is a compact subset of the complex plane $\mathbb{C}$, write $\partial K$ and $\text{iso}(K)$, respectively, for the topological boundary points and the isolated points of $K$.

We recall that $T \in \mathcal{L}(X, Y)$ is said to be bounded below if there is $k > 0$ for which $\|x\| \leq k\|Tx\|$ for all $x \in X$ and is said to be regular if there is $T' \in \mathcal{L}(Y, X)$ for which $T = TT'$. It is known that $T$ is regular if and only if $T(X)$ is closed and both $T^{-1}(0)$ and $T(X)$ are complemented and that

$$T \text{ regular and one-one } \Rightarrow T \text{ bounded below } \Rightarrow T(X) \text{ closed}$$

(cf. [2, 3]). Recall, also, that $T \in \mathcal{L}(X, Y)$ is said to be Fredholm if $T^{-1}(0)$ and $Y/T(X)$ are finite dimensional. If $T \in \mathcal{L}(X, Y)$ is Fredholm then the index of $T$ is defined by

$$\text{index}(T) = \dim T^{-1}(0) - \dim Y/T(X).$$

If $T \in \mathcal{L}(X)$ then the hyperrange of $T$ is the subspace

$$T^\infty(X) = \bigcap_{n=1}^{\infty} T^n(X).$$

We begin with a modification of [2, Theorem 7.8.3]:

Received by the editors May 9, 1991.
1991 Mathematics Subject Classification. Primary 47A10.
Key words and phrases. Punctured neighborhood theorem, regular operators, spectrum.
Research supported in part by a grant from ADD in 1991.
Lemma 1. Let $X$ be a normed space and $T \in \mathcal{L}(X)$. If the intersection $T^{-1}(0) \cap T^k(X)$ is finite dimensional for some $k$ then

$$T(T^\infty(X)) = T^\infty(X).$$

(1.1)

If $S \in \mathcal{L}(X)$ is invertible and commutes with $T$, then

$$\quad (T - S)^{-1}(0) \subseteq T^\infty(X).$$

(1.2)

Proof. The proof of equality (1.1) is taken straight from a slight modification of the proof of [2, (7.8.3.2)], which works with the stronger assumption $\dim T^{-1}(0) < \infty$. The inclusion (1.2) is just the inclusion [2, (7.8.3.4)].

Our main theorem is a generalization of the “punctured neighborhood theorem.”

Theorem 2. If $T \in \mathcal{L}(X)$ is regular on a Banach space $X$, with finite-dimensional intersection $T^{-1}(0) \cap T(X)$, and if $S$, $S'$ are invertible, commute with $T$ and have sufficiently small norm, then

$$\dim (T - S')^{-1}(0) = \dim (T - S)^{-1}(0)$$

and

$$\dim X/(T - S')X = \dim X/(T - S)X.$$  

(2.1)

(2.2)

Proof. Suppose $T \in \mathcal{L}(X)$ is regular and $T^{-1}(0) \cap T(X)$ is finite dimensional. We begin by showing $T^\infty(X)$ is complete. To do this, define $S_1: X/T^{-1}(0) \rightarrow X$ by setting

$$S_1(x + T^{-1}(0)) = Tx \in X \quad \text{for each } x \in X.$$  

Then, by (0.1) $S_1$ is bounded below. Our assumption also gives (with the aid of [5, Lemma 1]) that the subspace $T(X) + T^{-1}(0)$ is closed in $X$. Further, we can find a closed subspace $W \subseteq T(X)$ for which

$$T(X) + T^{-1}(0) = W + T^{-1}(0) \quad \text{and} \quad T(X) = W \oplus (T^{-1}(0) \cap T(X)).$$

We can then regard $W + T^{-1}(0) = \{w + T^{-1}(0) : w \in W\}$ as a closed subspace of $X/T^{-1}(0)$. If we define $S_2: W + T^{-1}(0) \rightarrow X$ by setting

$$S_2(w + T^{-1}(0)) = Tw \in X \quad \text{for each } w \in W,$$

then $S_2$ is also bounded below (see [2, (3.11.1.2)]). Since $W + T^{-1}(0)$ is complete, it follows from (0.1) that $S_2(W + T^{-1}(0)) = T(W) = T^2(X)$ is closed in $X$. Inductively, we have that $T^n(X)$ is closed in $X$ for each $n \in \mathbb{N}$, hence so is $T^\infty(X)$; therefore, $T^\infty(X)$ is complete. We write $U^\wedge: T^\infty(X) \rightarrow T^\infty(X)$ for the operator induced by $U \in \text{comm}(T)$, where $\text{comm}(T)$ is the “commutant” of $T$ in $\mathcal{L}(X)$. Then, since $(T^\wedge)^{-1}(0) = T^{-1}(0) \cap T^\infty(X) \subseteq T^{-1}(0) \cap T(X)$ and, by (1.1), $T^\wedge$ is onto, it follows that $T^\wedge$ is Fredholm. If $S$ has sufficiently small norm then $(T - S)^\wedge$ is also Fredholm because the Fredholm operators on a Banach space form an open set. We now claim that

$$\dim (T - S)^{-1}(0) = \dim (T - S)^\wedge^{-1}(0) = \text{index}(T - S)^\wedge = \text{index}(T^\wedge).$$

(2.3)

The first equality comes from (1.2), the second equality comes from the fact that, by the first equality and (1.1), $(T - S)^\wedge$ is onto, and the third equality comes from the continuity of the Fredholm index. Since the right-hand side of
(2.3) is independent of $S$, equality (2.1) follows. Also applying the "classical" punctured neighborhood theorem of $T - S$ gives the equality (2.2).

Our proof of Theorem 2 closely follows the original argument of Harte [2, Theorem 7.8.4], which assumes $T$ is Fredholm.

The following result is an improvement of [5, Theorem 2]:

**Corollary 3.** If $T \in \mathcal{L}(X)$ is regular on a Banach space $X$, with finite-dimensional intersection $T^{-1}(0) \cap T(X)$, then there is implication

\[(3.1) \quad 0 \in \partial \sigma(T) \Rightarrow 0 \in \text{iso } \sigma(T).\]

**Proof.** Apply Theorem 2 to $T - S$ with $S = \mu I$ and $0 < |\mu| < \varepsilon$; then $\dim(T - \lambda I)^{-1}(0)$ and $\dim X/(T - \lambda I)X$ are constant on a punctured neighborhood of 0. If $0 \in \partial \sigma(T)$ then it follows that for some $\theta$ with $0 < \theta < \varepsilon$, $\dim(T - \lambda I)^{-1}(0) = \dim X/(T - \lambda I)X = 0$ for $0 < |\lambda| < \theta$, which says that $0 \in \text{iso } \sigma(T)$.

Recall that $T \in \mathcal{L}(X)$ is said to be relatively almost open if its truncation $\hat{T}: X \to T(X)$ is almost open (cf. [2, 4]). If $T \in \mathcal{L}(X)$ for a Banach space $X$ then by the open mapping theorem we have

\[(3.2) \quad T \text{ relatively almost open } \iff T(X) \text{ closed.}\]

In the context of a Hilbert space we can simplify Corollary 3. In a sense, the following result is an improvement of [6, Theorem 1].

**Corollary 4.** If $X$ is a Hilbert space and $T \in \mathcal{L}(X)$ is relatively almost open, with finite-dimensional intersection $T^{-1}(0) \cap T(X)$, then there is implication

\[(4.1) \quad 0 \in \partial \sigma(T) \Rightarrow 0 \in \text{iso } \sigma(T).\]

**Proof.** If $T \in \mathcal{L}(X)$ for a Hilbert space $X$ then both $T^{-1}(0)$ and $\text{cl } T(X)$ are always complemented; thus we have

\[(4.2) \quad T \text{ regular } \iff T(X) \text{ closed; therefore, (3.1) together with (3.2) and (4.2) gives (4.1).}\]

**Acknowledgment**

The author would like to express his thanks to the referee whose suggestions led to an improvement of the paper.

**References**


**Department of Mathematics, Sung Kyun Kwan University, Suwon 440-746, Korea**