WHEN IS $F[x, y]$ A UNIQUE FACTORIZATION DOMAIN?

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Abstract. Although the commutative polynomial ring $F[x, y]$ is a unique factorization domain (UFD) and the free associative algebra $F(x, y)$ is a similarity-UFD when $F$ is a (commutative) field, it is shown that the polynomial ring $F[x, y]$ in two commuting indeterminates is not a UFD in any reasonable sense when $F$ is the skew field of rational quaternions.

As anyone who has had a course in abstract algebra knows, the polynomial ring $F[x, y]$ in two variables over a field $F$ is a unique factorization domain (UFD). In generalizing to the noncommutative case there are at least two natural possibilities to consider.

First we take $x$ and $y$ to be noncommutative while the field of coefficients remains commutative. Specifically, we consider the free associative algebra $R = F(x, y)$. It can be shown that this ring is a similarity-UFD. Thus for two factorizations

$$p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m,$$

where the $p_i$ and $q_i$ are irreducibles in $R$, $n = m$, and there is a permutation $\sigma$ of the subscripts such that $p_i$ and $q_{\sigma(i)}$ are similar, which means $R/p_i R \cong R/q_{\sigma(i)} R$ as $R$-modules (see [1, p. 9]). For the second case we assume that $x$ and $y$ commute but take $F$ to be a skew field. It is our purpose to show that in this situation factorization into irreducibles is not generally unique in any reasonable sense.

If $R$ is any (not necessarily commutative) integral domain then an atom (or irreducible) in $R$ is a nonzero nonunit that has no proper factors. Recall (see [2, p. 159]) that elements $a$ and $a'$ are similar (i.e., $R/a R \cong R/a' R$ as $R$-modules) if and only if there exists $b \in R$ such that

$$a R + b R = R, \quad a R \cap b R = ba' R.$$

If, in addition, $a$ is a central atom then $ba \in a R \cap b R = ba' R$ so that $a R \subseteq a' R \subseteq R$, which implies $a R = a' R$. This establishes the following result.

Proposition 1. Let $R$ be any (not necessarily commutative) integral domain with similar elements $a$ and $a'$. If $a$ is a central atom then $a$ and $a'$ are (right) associates.
In the remainder of this note we shall let $R = F[x, y]$ where $F = \mathbb{Q}(1, i, j, k)$ is the quaternion algebra over the field $\mathbb{Q}$ of rational numbers. We refer to [4] for any needed results on quaternions. Thus $F$ is a skew field in which each element $a$ has the form

$$a = a_0 + a_1 i + a_2 j + a_3 k$$

where $a_i$ is in the center $C(F)$ of $F$, which is the field $\mathbb{Q}$. Similarly, each $f \in R$ has the form

$$f = f_0 + f_1 i + f_2 j + f_3 k$$

where $f_i$ is in the center $C(R)$ of $R$, which is $\mathbb{Q}[x, y]$. The conjugate of $f$ is defined to be

$$\bar{f} = f_0 - f_1 i - f_2 j - f_3 k.$$ 

Note that

$$ff = f_0^2 + f_1^2 + f_2^2 + f_3^2 \in \mathbb{Q}[x, y].$$

Furthermore, conjugation is an antiautomorphism of $R = F[x, y]$. We refer to irreducible polynomials as atoms and to nonconstant central polynomials that have no proper central factors as $C$-atoms.

In the polynomial ring $F[x]$ the relationship between atoms and $C$-atoms is easily determined. We shall describe the situation for the polynomial ring $K[x]$ where $K$ is the quaternion algebra over any field of characteristic $\neq 2$ and over which the equation

$$a^2 + b^2 + c^2 + d^2 = 0$$

has only the trivial solution (these two conditions ensure that $K$ is a noncommutative field [4, p. 301]). Conjugation is defined and is an antiautomorphism of $K[x]$, analogous with the above description. Recall that $K[x]$ is a similarity-UFD [1].

**Proposition 2.** Let $K$ be the quaternion algebra just described. Let $f$ be a polynomial in $K[x]$ that is not associated to a central polynomial. Then $f$ is an atom if and only if $ff$ is a $C$-atom.

**Proof.** Assume that $f$ is an atom. Since $f \to \bar{f}$ is an antiautomorphism of $K[x]$, $\bar{f}$ is also an atom. Suppose $f\bar{f} = gh$ where $g$ and $h$ are central nonunits. Unique factorization in $K[x]$ shows that both $g$ and $h$ are atoms and $f$ is similar to $g$ or to $h$. Proposition 1 then shows that $f$ is associated to $g$ or to $h$ and this contradicts the hypothesis. Conversely, if $f\bar{f}$ is a $C$-atom and $f = rs$ where $r, s \in K[x]$ then $f\bar{f} = r\bar{f}ss$, which forces either $r\bar{f}$ or $ss$ to be a unit. Thus either $r$ or $s$ is a unit and $f$ is an atom.

We shall now show that Proposition 2 is not valid for polynomials in two variables. Consider the polynomial ring $R = F[x, y]$ over the rational quaternions and the polynomial

$$f = (x^2y^2 - 1) + (x^2 - y^2)i + 2xyj.$$ 

Then $f\bar{f}$ is not a $C$-atom; we have

$$f\bar{f} = (x^2y^2 - 1)^2 + (x^2 - y^2)^2 + 4x^2y^2 = (x^4 + 1)(y^4 + 1).$$

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which is a product of two $C$-atoms. Note that the unique factorization
\[ x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1) \]
that is carried out over the reals shows that the first factor in (1) is a $C$-atom in $R$ (i.e., irreducible over the rationals).

We claim that $f$ is an atom. If we write
\[ f = ax^2 + bx + c \]
where
\[ a = y^2 + i, \quad b = 2yj, \quad c = -1 - y^2i, \]
and if we view $f$ in $F(y)[x]$, then $f \bar{f}$ is a $C$-atom [3, p. 339], and so $f$ is an atom there by Proposition 2. Thus the only possible proper factorization of $f$ in $F[x, y]$ is of the form
\[ f = r(y)s(x) \quad \text{(or} \quad f = s(x)r(y)). \]
But then $r(y)$ must be a common left (or right) factor of $a$, $b$, and $c$, and so $r(y)\bar{r}(y)$ must be a common factor of
\[ aa - cc = y^4 + 1 \quad \text{and} \quad bb = 4y^2. \]
Since this is possible only if $r(y)\bar{r}(y)$ is a unit, we conclude that $f$ can have no proper factorization; that is, $f$ is an atom. Thus Proposition 2 fails in this case.

Equation (1) may be written
\[ f \bar{f} = (x^2 + i)(x^2 - i)(y^2 + i)(y^2 - i), \]
showing a product of two atoms equal to a product of four atoms (the latter are atoms by Proposition 2). Atomic factorization in $R$ is not unique.

We shall show that the behavior of degree 1 atoms in $R$ is more predictable than that exhibited in (2).

**Proposition 3.** Let $f$ be a linear polynomial in $R = F[x, y]$ that is not associated to a central polynomial. Then $f \bar{f}$ is a $C$-atom.

**Proof.** Let $f = ax + by + c$. If either $a$ or $b$ is 0 then $f \in F[y]$ or $F[x]$, respectively, so $f \bar{f}$ is a $C$-atom by Proposition 2. Suppose now that both $a$ and $b$ are nonzero and $f \bar{f} = rs$ where $r$, $s \in C(R)$. Viewing this equation in $F(x)[y]$ we see that $f \bar{f}$ must be a $C$-atom in $F(x)[y]$ so that either $r$ or $s$ is a unit, that is, a member of $F(x)$. Let us assume (without loss in generality) that it is $r$, so that $r \in F[x]$. Similarly we find that $r$ or $s$ is a member of $F(y)$. If $r$ is a unit in $F(y)$ then $r \in F[y] \cap F[x] = F$ and we are finished. If $s$ is a unit in $F(y)$ then $s \in F[y]$. However, the equation $f \bar{f} = r(x)s(y)$ is not possible when $a \neq 0$ and $b \neq 0$. Thus $f \bar{f}$ is a $C$-atom in $R$.

**Corollary.** Any product of linear polynomials in $R = F[x, y]$ is unique in the sense that if
\[ p_1p_2\cdots p_n = q_1q_2\cdots q_m \]
where the $p_i$, $q_i$ are degree one polynomials then $n = m$ and there is a permutation $\sigma$ of the subscripts such that $p_i\bar{p}_i$ are $q_{\sigma(i)}\bar{q}_{\sigma(i)}$ are associates. If $p_i$ is associated to a central polynomial then $p_i$ and $q_{\sigma(i)}$ are associates.

**Proof.** The result follows from the equation
\[ p_1\bar{p}_1p_2\bar{p}_2\cdots p_n\bar{p}_n = q_1\bar{q}_1q_2\bar{q}_2\cdots q_m\bar{q}_m \]
of \( C \)-atoms in the unique factorization domain \( \mathbb{Q}[x, y] \). Note that if \( p_i \) is associated to a \( C \)-atom then \( p_i \overline{p}_i \) is associated to the square of that \( C \)-atom; otherwise, \( p_i \overline{p}_i \) is itself a \( C \)-atom.

It is difficult to improve on the corollary: (2) shows a product of two atoms equal to a product of four atoms. Thus when \( F \) is the skew field of rational quaternions, \( R = F[x, y] \) cannot be considered to be a unique factorization domain in any reasonable sense.

**References**


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