DIFFERENTIATION OF ZYGMUND FUNCTIONS

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Abstract. The “little-\(o\) Zygmund class” \(\lambda^*\) contains a nowhere-differentiable function.

0. Introduction

A classical result due originally to Rajchman and then improved by Zygmund [ZY, p. 43] states that if \( f \in \lambda^*(T) \) and \( f \) is real valued then \( f \) must be differentiable on a dense subset of \( T \). This implies that if \( F \in \mathcal{B}_o \) (the “little-\(o\) Bloch space”) then \( \text{Re}(F) \) must possess a radial (and hence nontangential) limit at each point of a dense subset of the boundary.

Somewhat more recently, it was shown [GHP, Theorem 2] that \( F \) itself must have a radial limit at each point of a dense subset of the boundary, if \( F \in \mathcal{B}_o \). As noted in [GHP], this would follow from the result of Rajchman and Zygmund if the latter were true for a general (complex-valued) element of \( \lambda^* \), but this question has been open. In this note we show that there exists an \( f \in \lambda^* \) that is nowhere differentiable and that, in fact, satisfies a Hölder condition of order one at no point.

It turns out that the existence of a nowhere differentiable \( f \in \lambda^* \) is also one of various results in [MAK2], including the fact that if \( f \in \lambda^* \) and is either real-valued or extends to a function holomorphic in the disc then the set of points where \( f \) is differentiable must have Hausdorff dimension 1. (The results in [MAK2] are proved in more detail in [MAK1], in particular, cf. [MAK1, Theorem 5.5].) It seems that the extremely simple argument below may nonetheless be of some independent interest: If \( u \) is an appropriate (real-valued) lacunary trigonometric series then \( u \in \lambda^* \) and \( u \) is differentiable only on a set of measure zero. Now one may construct \( v \in \lambda^* \) so that \( f = u + iv \) is nowhere differentiable (in particular, we do not require the main technical device in [MAK2]—a characterization of the dyadic martingales arising from elements of \( \lambda^* \)).

1. Theorem

The notation \( \lambda^*(T) \) refers to the “little-\(o\) ” Zygmund class on the unit circle \( T \) : we write \( f \in \lambda^*(T) \) if \( f \) is a continuous (complex-valued) function on \( T \).
and
\[ \lim_{h \to 0} h^{-1}|f(e^{it-h}) - 2f(e^{it}) + f(e^{it+h})| = 0, \]
uniformly in \( t \) (the functions in \( \mathcal{X}^* \) are called “smooth functions” in [ZY]).

We set
\[ M_f(t) = \sup_{h>0} h^{-1}|f(e^{it+h}) - f(e^{it})|, \]
so that \( f \) satisfies a Hölder condition of order 1 at \( e^{it} \) if and only if \( M_f(t) < \infty \).

**Theorem.** There exists \( f \in \mathcal{X}^*(T) \) such that \( M_f(t) \equiv \infty \).

We will set \( f = u + iv \), where \( u \in \mathcal{X}^* \) is a (real-valued) lacunary series with \( Mu = \infty \) a.e. It is impossible to achieve \( Mu \equiv \infty \) here, but the following proposition will provide us with a real-valued function \( v \in \mathcal{X}^* \) such that \( Mv = \infty \) at every point of the set where \( Mu < \infty \).

**Proposition.** Suppose \( E \subset T \) is an \( F_\alpha \) of (Lebesgue) measure zero. Then there exists a real-valued \( v \in \mathcal{X}^*(T) \) such that \((d/dt)v(e^{it}) = \infty \) for every \( t \in E \).

This will follow from the following lemma. The notation \( \text{VMO}(T) \) refers to the space of functions of vanishing mean oscillation, as usual.

**Lemma.** Suppose \( E \) is as in the proposition. There exists \( \phi \in \text{VMO}(T) \) such that \( \phi \geq 0 \) on \( T \) and \( \lim_{s \to t} \phi(e^{is}) = \infty \) for every \( e^{it} \in E \).

**Proof.** If we can prove the lemma for compact \( E \) then the general case follows because \( \phi \geq 0 \). Suppose \( E \subset T \) is a compact set of measure zero.

This implies that \( E \) is a peak set for the disc algebra: there exists a function \( g \) that is holomorphic in the unit disc \( D \) and continuous on \( \overline{D} \), such that \( g(e^{it}) = 1 \) for \( e^{it} \in E \), while \( |g(z)| < 1 \) for \( z \in \overline{D} \setminus E \).

Now let \( \Omega = \{ x + iy : x > 1, |y| < 1/x \} \) and let \( \psi : D \to \Omega \) be holomorphic and surjective. A theorem of Carathéodory shows that \( \psi \) extends to a homeomorphism \( \overline{\psi} : \overline{D} \to \overline{\Omega} \), where \( \overline{\Omega} \) denotes the closure of \( \Omega \) on \( S \), the Riemann sphere; we may take \( \overline{\psi}(1) = \infty \).

Thus \( G = \overline{\psi} \circ g : \overline{D} \to S \) is continuous. Let \( \phi = \text{Re}(G) \). Then \( \phi \) (restricted to \( T \)) is a continuous map from \( T \) to \([0, \infty]\) such that \( \phi(e^{it}) = \infty \) for \( e^{it} \in E \). We only need to show that \( \phi \in \text{VMO} \), but \( \phi \in \text{VMO} \) because \( \phi \) is the harmonic conjugate of a continuous function: The point to our choice of \( \Omega \) was that \( \text{Im}(z) \to 0 \) as \( z \) tends to \( \infty \) within \( \Omega \), and this shows that \( \text{Im}(G) \in C(T) \). \( \square \)

**Proof of the proposition.** Given an \( F_\alpha \) set \( E \subset T \) of measure zero, choose \( \phi \) as in the lemma. Now define \( \phi_1 = \phi - c \), where \( c = (2\pi)^{-1} \int_0^{2\pi} \phi(e^{it}) \, dt \), and let \( v \) be an absolutely continuous function such that \( (d/dt)v(e^{it}) = \phi_1(e^{it}) \) almost everywhere. It follows that \( (d/dt)v(e^{it}) = \infty \) for \( t \in E \), while the fact that \( \phi \in \text{VMO} \) implies that \( v \in \mathcal{X}^* \). \( \square \)
Proof of the theorem. Choose a sequence $a_j \geq 0$ with $\lim_{j \to \infty} a_j = 0$ but $\sum_{j=1}^{\infty} a_j^2 = \infty$, and set

$$u(e^{it}) = \sum_{j=1}^{\infty} 2^{-j} a_j \cos(2^j t).$$

Now the fact that $a_j \to 0$ shows that $u \in \lambda^*$ [ZY, Theorem 4.10, p. 47], while $\sum_{j=1}^{\infty} a_j^2 = \infty$ shows that $Mu(e^{it}) = \infty$ for almost all $t$. This will be "clear" to readers with some experience dealing with lacunary series; a proof is already at least implicit in [ZY]:

Let $d_N(t) = -\sum_{j=1}^{N} a_j \sin(2^j t)$. Then it is well known that $(d_N(t))$ is unbounded for almost every value of $t$ [ZY, Theorem 6.4, p. 203 and Remark (c), p. 205]. But it is easy to obtain a uniform upper bound on the quantity

$$h_N^{-1}[u(e^{i(t+h_N)}) - u(e^{it})] - d_N(t)$$

if $h_N = 2^{-N} \pi$, so that $Mu = \infty$ at any point where $(d_N)$ is unbounded.

Now let $E = \{ e^{it} : Mu(e^{it}) < \infty \}$. We have just seen that $E$ has measure zero. Continuity of $u \in \lambda^*$ shows that $\{Mu \leq j\}$ is closed for $j = 1, 2, \ldots$, so that $E$ is an $F_\sigma$. Choose $v$ as in the proposition and let $f = u + iv$. Then $f \in \lambda^*$ and $Mf \equiv \infty$. □

References


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