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THE SUBALGEBRA OF $L^1(AN)$ GENERATED BY THE LAPLACIAN

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ABSTRACT. We prove that for the Iwasawa AN groups corresponding to complex semisimple Lie groups, the subalgebra of $L^1(AN)$ associated to a distinguished laplacian is isomorphic with the algebra of integrable radial functions on R^n.

In [1] Cowling et al. derived a formula for the heat semigroup generated by a distinguished laplacian on a large class of Iwasawa AN groups and proved that the maximal function constructed from the semigroup is of weak type $(1, 1)$.

In this paper we show that in the case of the AN groups corresponding to complex semisimple Lie groups the results of [1] can be strengthened once we notice that the subalgebra of $L^1(AN)$ associated to a distinguished laplacian is isomorphic with the algebra of integrable radial functions on R^n. This implies that the maximal operators associated to the Riesz means are of weak type $(1, 1)$. Also the functional calculus for the laplacian on R^n can be transferred to the distinguished laplacian on AN. This seems to be the first construction of a nonanalytic functional calculus on groups of exponential growth.

Let G denote a connected, complex semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Denote by θ a Cartan involution of $\mathfrak{g}$, and write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

for the associated Cartan decomposition. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$; this determines a root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha,$$

Λ denoting the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Corresponding to a choice of the ordering of the roots, we have an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}.$$

Let $G = ANK$ be the corresponding Iwasawa decomposition of G. A distinguished laplacian on AN can be constructed as follows. Let $\pi\colon \mathfrak{g} \to \mathfrak{p}$ be the projection (defined by the Cartan decomposition). We define a positive definite form $\widetilde{B}$ on $\mathfrak{a} \oplus \mathfrak{n}$ setting $\widetilde{B}(X, Y) = B(\pi X, \pi Y)$ where B is the Killing form

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on $\mathfrak{g}$. Put $n = \dim(AN)$. Choose an orthonormal (with respect to $\widetilde{B}$) basis in $\mathfrak{a} \oplus \mathfrak{n}$, say $\{X_1, \dots, X_n\}$.

Define the laplacian Δ by setting

$$\Delta f(x) = \sum_{j=1}^{n} \left(\frac{d}{dt}\right)^2 f(x \exp(tX_j))|_{t=0}.$$

Denote by $|x|$ the riemannian distance between e and x corresponding to the left-invariant riemannian structure induced by $\widetilde{B}$. Let ϕ_0 be the distinguished spherical function (restricted to AN)—ϕ_0 may be defined as the unique function that is radial, i.e., we have $\phi_0(x) = \psi(|x|)$ for some ψ, and that satisfies the equation $\Delta(\delta^{-1/2}\phi_0) = 0$ where δ is the modular function of AN.

The heat kernel p_t corresponding to Δ is given by the formula (see [1])

$$p_t(x) = C_0 t^{-n/2} \varphi_0(x) \delta^{-1/2}(x) e^{-|x|^2/(4t)}.$$

Let A be the subalgebra of $L^1(AN)$ (with respect to right-invariant Haar measure) generated by p_t, $t > 0$.

1. **Theorem.** *The operator T given by the formula*

$$(Tf)(x) = C_0 (4\pi)^{n/2} \varphi_0(x) \delta^{-1/2}(x) f(|x|)$$

is an isometric algebra isomorphism between $L^1_{\mathrm{rad}}(R^n)$ and A.

Proof. Let V denote the vector space of all linear combinations of q_t, $t > 0$, where q_t is the heat kernel on R^n. By the formula above, T restricted to V is an isomorphism of algebras. Moreover, for all $f \in V$ we have

$$\int_{AN} Tf = \int_{R^n} f.$$

On the other hand, if we denote by E the space $L^1_{\mathrm{rad}}(R^n, e^{C|x|}\,dx)$ for sufficiently large C, then T is continuous from E into $L^1(AN)$. Since V is dense in E, it follows that $T(E) \subset A$ and for all $f \in E$ we have

$$\int_{AN} Tf = \int_{R^n} f.$$

Decomposing a function f in E into its positive and negative part (which of course belong to E) we have

$$\|f\|_{L^1_{\mathrm{rad}}} = \int_{R^n} f_+ + \int_{R^n} f_- = \int_{AN} Tf_+ + \int_{AN} Tf_- = \|Tf\|_{L^1(AN)}.$$

Now the closure of $T|_V$ is an isometry of $L^1_{\mathrm{rad}}(R^n)$ with A. Obviously this closure is equal to T, which ends the proof.

2. **Corollary.** *The spectrum of Δ on $L^p(AN)$, $\infty > p \geq 1$, does not depend on p and equals R_+.*

Let $R_\alpha(x) = (1 - x)_+^\alpha$. We put

$$S_\alpha f = \sup_{t>0} |R_\alpha(-t\Delta) f|$$

where $R_\alpha(-t\Delta)$ is well defined by the spectral theorem.

3. **Corollary.** *For $\alpha > (n-1)/2$ the maximal operator S_α for Riesz means of order α is of weak type $(1, 1)$.*

Proof. Let $Mf = \sup_{t>0} t^{-1} \int_0^t |f| * p_s \, ds$. On R^n, S_α is majorized by M. T preserves pointwise majorization of kernels, so this majorization holds also on AN. But M is of weak type $(1, 1)$ by the Dunford-Schwartz maximal ergodic theorem.

4. **Corollary.** *Let $k > (n-1)/2$, $t > 0$, $\varepsilon > 0$, $F \in C^k(R)$, and*

$$\sup_{\lambda>0}(1+\lambda)^{l+\varepsilon}|F^{(l)}(\lambda)| < \infty, \qquad l = 0, \dots, k.$$

Then the operator $F(-t\Delta)$ (defined by the spectral theorem on $L^1 \cap L^2$) is bounded on $L^1(AN)$.

References

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