INTEGRABLE MEAN PERIODIC FUNCTIONS
ON LOCALLY COMPACT ABELIAN GROUPS

INDER K. RANA AND K. GOWRI NAVADA

(Communicated by J. Marshall Ash)

Abstract. Let G be a locally compact abelian group with a Haar measure \( \lambda_G \). A function \( f \) on \( G \) is said to be mean-periodic if there exists a nonzero finite regular measure \( \mu \) of compact support on \( G \) such that \( f * \mu = 0 \). It is known that there exist no nontrivial integrable mean periodic functions on \( \mathbb{R}^n \). We show that there exist nontrivial integrable mean periodic functions on \( G \) provided \( G \) has nontrivial proper compact subgroups. Let \( f \in L_1(G) \) be mean periodic with respect to a nonzero finite measure \( \mu \) of compact support. If \( \mu(G) \neq 0 \) and \( \lambda_G(\text{supp}(\mu)) > 0 \), then there exists a compact subgroup \( K \) of \( G \) such that \( f * \lambda_K = 0 \), i.e., \( f \) is mean periodic with respect to \( \lambda_K \), where \( \lambda_K \) denotes the normalized Haar measure of \( K \). When \( G \) is compact, abelian and metrizable, we show that there exists continuous (hence integrable and almost periodic) functions on \( G \) that are not mean periodic.

1. Introduction

The theory of mean periodic functions on the real line \( \mathbb{R} \) was initiated by Delsarte [2, 3] and were analyzed in detail by Schwartz [11], Kahane [7], and Malgrange [8] on \( \mathbb{R}^n \) (see also Berenstein and Taylor [1] for mean periodic functions on \( \mathbb{R}^n \) and other references). Ehrenpreis and Mautner [5] started the study of mean periodic functions on the group \( \text{SL}(2, \mathbb{R}) \). It is well known that there does not exist nontrivial integrable mean periodic functions on \( \mathbb{R}^n \). However, Sitaram [12] has given an example of a nontrivial integrable mean periodic function on \( \text{SL}(2, \mathbb{R}) \). The aim of this note is to analyze mean periodic integrable functions on locally compact abelian groups.

Let \( G \) be a locally compact abelian group with a Haar measure \( \lambda_G \). We shall denote by \( M(G) \) the set of all finite regular measures on \( G \), and \( M_c(G) \) will denote the set of all \( \mu \in M(G) \), \( \mu \) with compact support. \( L_1(G) \) denotes the space of all integrable functions on \( G \).

1.1. Definition. We call \( f \in L_1(G) \) mean periodic if there exists \( \mu \in M_c(G) \), \( \mu \neq 0 \), such that \( f * \mu = 0 \); here \( * \) denotes the convolution operation. We shall also say \( f \) is mean periodic with respect to \( \mu \).
1.2. Example. Let $K$ be any compact abelian group, and let $\hat{K}$ denote the dual group of $K$. Let $\gamma \in \hat{K}$ be any nontrivial character. Then for $x \in K$,

$$(\gamma \ast \lambda_K)(x) = \int \langle x - y, \gamma \rangle d\lambda_K(y) = \langle x, \gamma \rangle \int \langle \overline{y}, \overline{\gamma} \rangle d\lambda_K(y).$$

Also $\int_K \langle x - y, \gamma \rangle d\lambda_K(x) = \int_K \langle \overline{y}, \overline{\gamma} \rangle d\lambda_K(y)$. Since $\gamma$ is nontrivial, we have $\int_K \langle x - y, \gamma \rangle d\lambda_K(x) = 0$, i.e., $\gamma \ast \lambda_K = 0$.

Thus every nontrivial $\gamma \in \hat{K}$ is mean periodic. Also if $f$ is any constant function on $K$ and $K$ has at least two distinct points $k_1, k_2 \in K$, then $f \ast (\delta_{k_1} - \delta_{k_2}) = 0$, where $\delta_k$ denotes the unit mass at $k$.

1.3. Example. Let $T$ denote the circle group and $f(x) = x$, $x \in T$. We have already seen in Example 1.2 that $f$ is mean periodic with respect to $\lambda_T$, the normalized Haar measure on $T$. Let $\nu$ denote the measure on $T$ given by $d\nu(x) = 1/xd\lambda_T(x)$. Then it is easy to see that $f \ast \nu = 0$, i.e., $f$ is mean periodic with respect to $\nu$.

1.4. Example. Let $G$ be a locally compact abelian group such that $G$ has a nontrivial proper compact subgroup $K$. Let $\Lambda = \{\gamma \in \hat{G} | \langle x, \gamma \rangle = 0 \text{ for all } x \in K\}$ denote the annihilator of $K$. Then $\Lambda$ is a proper closed subgroup of $\hat{G}$. Let $\gamma_0 \in \hat{G} \setminus \Lambda$. Choose a neighbourhood $U$ of $\gamma_0$ and $V$ of $\Lambda$ such that $U \cap V = \emptyset$. Then $\gamma_0 \in U \subseteq V^c$. Now choose $f \in L_1(G)$ such that $\hat{f}(\gamma_0) = 1$ and $\hat{f}(\gamma) = 0$ for every $\gamma \in U^c$ (see [11, Theorem 2.6.2]). Thus $\hat{f}(\gamma) = 0$ for every $\gamma \in \Lambda$, and hence $f \ast \lambda_K = 0$ (see [10, Theorem 2.7.4]). Thus, we have shown that there exists nontrivial $f \in L_1(G)$ that are mean periodic provided $G$ has nontrivial proper compact subgroups.

We next analyze integrable mean periodic functions on compact abelian groups.

2. INTEGRABLE MEAN PERIODIC FUNCTIONS ON COMPACT ABELIAN GROUPS

Let $K$ be a nontrivial compact abelian group. In view of Example 1.2, one may ask: is every $f \in L_1(K)$ mean periodic? We shall see that this is not so in general. However, every $f \in L_1(K)$ can be written as a sum of two integrable mean periodic functions on $K$.

2.1. Proposition. Let $K$ be a compact abelian group and $f \in L_1(K)$. Then $f = g + h$, where $g$, $h$ are mean periodic functions in $L_1(K)$.

Proof. Let $\lambda_K$ denote the normalized Haar measure of $K$. Let $h = f \ast \lambda_K$ and $g = f - h$. Then clearly $g \ast \lambda_K = 0$ and $h(x) = \int_K f d\lambda_K$ for every $x \in K$. Thus $g$, $h$ are mean periodic functions in $L_1(K)$ and $f = g + h$.

2.2. Remark. The above proposition says that $L_1(K) = S + C$, where $S$, $C \subseteq L_1(K)$ and every $f \in S \cup C$ is a mean periodic function. In fact, $C$ is the space of all constant functions on $K$. One can say something more about $S$. We claim that $S = \{g \in L_1(K) | g \ast \lambda_K = 0\}$ is a maximal ideal of $L_1(K)$. To see this we first note that $S$ is a closed subalgebra of $L_1(K)$. Next, if $\gamma_0 \in \hat{K}$ and $\gamma_0 \neq 1$, we choose a neighbourhood $W$ of $\gamma_0$ such that $1 \notin W$. Now we choose $g \in L_1(K)$ such that $\hat{g}(\gamma_0) = 1$ and $\hat{g}(\gamma) = 0$ for every $\gamma \in W^c$ (see [10, Theorem 2.6.2]). Thus, $\hat{g}(\gamma_0) = 1$ and $\hat{g}(1) = 0$. But then
INTEGRABLE MEAN PERIODIC FUNCTIONS

407

$(g * \lambda_K)(x) = \int_K g(x - y) d\lambda_K(y) = \int_K g(y) d\lambda_K(y) = \hat{g}(1) = 0$. Thus, for $g \in S$, $\hat{g}$ separates points of $\hat{K}$. Hence by a corollary [10, p. 232] either $S = L_1(K)$ or $S$ is a maximal ideal of $L_1(K)$. Since $S \neq L_1(K)$, $S$ is a maximal ideal of $L_1(K)$.

We show next that not every $f \in L_1(K)$ is mean periodic.

2.3. Proposition. Let $K$ be a compact metrizable abelian group and let $\hat{K}$ be its dual group. Let $f = \sum_{\gamma \in \hat{K}} a_{\gamma} \gamma$, where $a_{\gamma} \in \mathbb{C}$ and the series converges uniformly on $K$. Then there exists a nonzero finite measure $\mu$ on $K$ such that $f * \mu = 0$ iff $a_{\gamma} = 0$ for at least one $\gamma \in \hat{K}$.

Proof. Clearly $a_{\gamma} = \hat{f}(\gamma)$ for every $\gamma \in \hat{K}$. Further $f * \mu = \sum_{\gamma \in \hat{K}} a_{\gamma} \hat{\mu}(\gamma) \gamma$. Now if $f * \mu = 0$, then for every $\gamma \in \hat{K}$, $0 = (f * \mu)(\gamma) = \hat{f}(\gamma) \hat{\mu}(\gamma) = a_{\gamma} \hat{\mu}(\gamma)$. Since $\mu \neq 0$, there exists at least one $\gamma_0 \in \hat{K}$ such that $\hat{\mu}(\gamma_0) \neq 0$. Thus, $a_{\gamma_0} = 0$. Conversely, if $a_{\gamma_0} = 0$ for some $\gamma_0 \in \hat{K}$, then $f * \mu = 0$ for $d\mu(x) = \gamma_0(x) d\lambda_K(x)$.

2.4. Corollary. Let $K$ be a compact metrizable abelian group and let $\hat{K}$ be its dual group. Let $f = \sum_{\gamma \in \hat{K}} a_{\gamma} \gamma$ where $0 \neq a_{\gamma} \in \mathbb{C}$ for every $\gamma \in \hat{K}$ and the series converges uniformly. Then $f$ is a continuous hence integrable (and almost periodic) function on $K$ and $f$ is not mean periodic.

2.5. Remark. Corollary 2.4 extends a remark of Kahane [7] that on $\mathbb{R}^n$ there exist almost periodic functions that are not mean periodic.

3. Integrable mean periodic functions

on locally compact abelian groups

Let $G$ be a locally compact abelian group. As seen in Example 1.4, if $G$ has a nontrivial compact subgroup $K$, then there exists $0 \neq f \in L_1(G)$ that is mean periodic. To be more precise, $f * \lambda_K = 0$, where $\lambda_K$ denotes the normalized Haar measure of $K$. We ask the question: if $f \in L_1(G)$ and is mean periodic, under what conditions can one say $f * \lambda_K = 0$ for some compact subgroup $K$ of $G$? The answer is given by the following

3.1. Theorem. Let $G$ be a locally compact abelian group and $f \in L_1(G)$ be mean periodic with respect to $0 \neq \mu \in M_c(G)$. If $\mu(G) \neq 0$ and $\lambda_G(\text{supp}(\mu)) > 0$, then there exists a compact subgroup $K$ of $G$ such that $f * \lambda_K = 0$.

To prove the theorem, we need some preliminary results. Let $G$ be a locally compact abelian group and $K$ a compact subgroup of $G$. Let $\lambda_G$, $\lambda_K$, and $\lambda_{G/K}$ denote the Haar measures of $G$, $K$, and $G/K$, respectively, such that $\lambda_K(K) = 1$ and such that for every $f \in L_1(G)$

$$
\int_G f d\lambda_G = \int_{G/K} \left( \int_K f(x + y) d\lambda_K(y) \right) d\lambda_{G/K}.
$$

Let $M(G)$ and $M(G/K)$ denote the space of all bounded regular measures on $G$ and $G/K$, respectively. Following [10, §2.7], we define

$$
T : M(G) \to M(G/K),
$$

$$(T \mu)(f) = \int_G (f \circ \Pi)(x) d\mu(x), \quad f \in C_0(G/K), \quad \mu \in M(G),$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
where \( \Pi: G \to G/K \) is the natural homomorphism and \( C_0(G/K) \) denotes the space of all continuous functions on \( G/K \) vanishing at \( \infty \). Given \( \mu \), \( T\mu \in M(G/K) \) is well defined and

\[
\int_G (f \circ \Pi)(x) \, d\mu(x) = \int_{G/K} f \, d(t\mu)
\]

for every bounded Borel function \( f \) on \( G/K \). The following properties of \( T \) are easy to check.

### 3.2. Proposition

(i) If \( \mu \in M(G) \) is such that either \( \mu(G) \neq 0 \) or \( \mu \neq 0 \) and \( \mu \) is \( K \)-invariant, then \( T(\mu) \neq 0 \).

(ii) If \( \mu \in M(G) \) has compact support, then so does \( T\mu \).

(iii) For \( f \in L_1(G) \), \( Tf \in L_1(G/K) \) and

\[
(Tf)(\Pi(x)) = \int_K f(x + k) \, d\lambda_K(k), \quad x \in G.
\]

(iv) For \( f \in L_1(G) \) and \( \mu \in M_c(G) \), \( T(f * \mu) = (Tf) * (\mu) \).

(v) If \( f \in L_1(G) \) is \( K \)-invariant and \( (Tf)(\Pi(x)) = 0 \) for every \( x \in G \), then \( f(x) = 0 \) for every \( x \in G \).

Our next lemma is motivated by the lemma of [9, p. 311]

### 3.3. Lemma

Let \( f \in L_1(G) \) be a continuous function and \( \mu \in M_c(G) \) be such that \( \mu(G) \neq 0 \), \( \lambda_G(\text{supp}(\mu)) > 0 \), and \( f * \mu = 0 \). Then there exists a compact subgroup \( K \) of \( G \) such that \( f * \lambda_K = 0 \).

**Proof.** Let \( E = \text{supp}(\mu) \). Then \( E \) is a compact subset of \( G \) and \( \lambda_G(E) > 0 \). Let \( G_0 \) be the subgroup of \( G \) generated by \( E \). Then \( G_0 \) is a locally compact compactly generated abelian group and, by the structure theorem (Hewitt and Ross [6]), \( G_0 \) is topologically isomorphic to \( \mathbb{R}^n \times \mathbb{Z}^k \times K \), where \( \mathbb{Z} \) denotes the integer group, \( n, k \), are nonnegative integers, and \( K \) is a compact abelian group. We assume without loss of generality that \( G_0 = \mathbb{R}^n \times \mathbb{Z}^k \times K \). To prove the lemma, we first assume that \( f \) is \( K \)-invariant, i.e., \( f(x + k) = f(x) \) for all \( x \in G, k \in K \). We shall show \( f * \lambda_K = f \equiv 0 \). Choose \( x_0 \in G \) arbitrarily and fix it. Let \( f_{x_0}(x) = f(x + x_0), \ x \in G \). Then \( f_{x_0} \) is a \( K \)-invariant continuous integrable function on \( G \) and \( f_{x_0} * \mu = 0 \). Since \( \mu \) is concentrated on \( G_0 \), this gives \( f_{x_0} * \mu = 0 \) on \( G_0 \). Using Proposition 3.2,

\[
(*) \quad T(f_{x_0} * \mu) = T(f_{x_0}) * T(\mu) = 0 \quad \text{on} \ \mathbb{R}^n \times \mathbb{Z}^k.
\]

Taking the Fourier transform, we have

\[
(**) \quad (Tf_{x_0})(\gamma) \cdot (T(\mu))(\gamma) = 0 \quad \text{for every} \ \gamma \in \mathbb{R}^n \times \mathbb{T}^k.
\]

Since \( \mu \in M_c(G) \) and \( \mu(G_0) \neq 0 \), again by Proposition 3.2, \( 0 \neq T\mu \in M_c(G_0/K) = M_c(\mathbb{R}^n \times \mathbb{Z}^k) \). Thus \( (T\mu)(\gamma) \neq 0 \) for almost all \( \gamma \in \mathbb{R}^n \times \mathbb{T}^k \).

Hence \( (**) \) gives \( (Tf_{x_0})(\gamma) = 0 \) for almost all \( \gamma \in \mathbb{R}^n \times \mathbb{T}^k \). Thus, \( (Tf_{x_0})(\vec{x}) = 0 \) for almost all \( \vec{x} \in \mathbb{R}^n \times \mathbb{Z}^k \). Since \( f_{x_0} \) is continuous, this implies \( Tf_{x_0}(\vec{x}) = 0 \) for all \( \vec{x} \in G_0/K = \mathbb{R}^n \times \mathbb{Z}^k \). Again, since \( f_{x_0} \) is \( K \)-invariant, by Proposition 3.2, this implies that \( f_{x_0}(x) \equiv 0 \). To prove the proposition in the general case, we consider the function \( f \) and apply the above case to conclude \( f * \lambda_K \equiv 0 \).

**Proof of Theorem 3.1.** Let \( f \in L_1(G) \) and \( \mu \in M_c(G) \) be such that \( \mu(G) \neq 0 \) and \( f \) \( K \)-invariant. Let \( \phi \in C_c(G) \), the space of continuous functions on \( G \) with
compact support. Then \((f * \mu) * \phi = (f * \phi) * \mu = 0\) on \(G\). Since \(f * \phi \in L_1(G)\) is a continuous function, by Lemma 3.3, \((f * \phi) * \lambda_K = 0\) for all \(\phi \in C_c(G)\). Thus, \((f * \lambda) * \phi = 0\) for all \(\phi \in C(G)\) and hence \(f * \lambda = 0\). This proves the theorem.

3.4. **Remark.** The condition that \(\mu(G) \neq 0\) in Theorem 3.1 can be replaced by a weaker condition: \(\mu * \lambda_K \neq 0\), where \(K\) is the maximal compact subgroup of \(G_0\), the subgroup of \(G\) generated by \(\text{supp}(\mu)\). To see this we note that in the proof of Lemma 3.3, the equation (*) can be replaced by

\[
T(f_{x_0} * \mu * \lambda_K) = T(f_{x_0}) * T\hat{\mu} = 0,
\]

where \(\hat{\mu} = \mu * \lambda_K\). Since \(\hat{\mu} \neq 0\) and is \(K\)-invariant, by Proposition 3.2, \(T(\hat{\mu}) \neq 0\) has compact support. Now proceeding as in Lemma 3.3, we will get Lemma 3.1 under the condition \(\mu * \lambda_K \neq 0\). Hence Theorem 3.1 will hold under the conditions: \(\mu * \lambda_K \neq 0\) and \(\lambda_G(\text{supp}(\mu)) > 0\).

In the next theorem, we give some equivalent versions of the conclusions of Theorem 3.1.

3.5. **Theorem.** Let \(G\) be a locally compact abelian group. Let \(f \in L_1(G)\) be mean periodic with respect to a measure \(\mu \in M_c(G)\), where \(\mu(G) \neq 0\) and \(\lambda_G(\text{supp}(\mu)) > 0\). Then the following hold:

(i) \(f\) is mean periodic with respect to the normalized Harr measure \(\lambda_K\) of some compact subgroup \(K\) of \(G\).

(ii) There exists a compact subgroup \(K\) of \(G\) such that \(\hat{f}(\gamma) = 0\) for all \(\gamma \in \text{Ann}(K)\).

(iii) There exists a compact subgroup \(K\) of \(G\) such that

\[
f \in \text{cl}\{g \in L_1(G) | \text{Ann}(K) \subset \text{Int} Z(g)\},
\]

where \(\text{cl}\) denotes the closure in \(L_1\)-norm, \(Z(g) = \{\gamma \in \hat{G} | \hat{g}(\gamma) = 0\}\), and \(\text{Int} Z(g)\) denotes the interior of \(Z(g)\).

**Proof.** Under the given conditions, clearly (i) holds by Theorem 3.1. We shall show (i) \(\Rightarrow\) (ii) \(\Leftrightarrow\) (iii). Implication (i) \(\Rightarrow\) (ii) is Theorem 2.7.4 of [10]. Suppose now (ii) holds. Since \(\Lambda = \text{Ann}(K)\) is a closed subgroup of \(G\), by Theorem 7.5.2(d) of [10], it follows that for every \(\varepsilon > 0\), there exists \(g \in L_1(G)\) such that \(\hat{g}\) has compact support disjoint from \(\Lambda\) and \(\|f - f * g\| < \varepsilon\). Further \(f * g \in I_0(\Lambda) = \text{cl}\{\phi \in L_1(G) | \Lambda \subset \text{Int} Z(\phi)\}\) by Theorem 7.5.2(a) of [10]. But then clearly \(f \in I_0(\Lambda)\). This proves (ii) \(\Rightarrow\) (iii). Also if \(f \in I_0(\Lambda)\), clearly \(\Lambda \subset Z(f)\) and thus (iii) \(\Rightarrow\) (ii).

**Acknowledgment**

The authors would like to thank the referee for the arguments presented in Proposition 2.3.

**References**