ASYMPTOTIC BEHAVIOR OF ALMOST-ORBITS OF NONLINEAR SEMIGROUPS OF NON-LIPSCHITZIAN MAPPINGS IN HILBERT SPACES

KOK-KEONG TAN AND HONG-KUN XU

(Communicated by Palle E. T. Jorgensen)

Abstract. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, $\mathcal{S} = \{T(t) : t \geq 0\}$ be a continuous nonlinear asymptotically nonexpansive semigroup acting on $C$ with a nonempty fixed point set $F(\mathcal{S})$, and $u : [0, \infty) \to C$ be an almost-orbit of $\mathcal{S}$. Then $\{u(t)\}$ almost converges weakly to a fixed point of $\mathcal{S}$, i.e., there exists an element $y$ in $F(\mathcal{S})$ such that

$$\text{weak-}\lim_{t \to \infty} \frac{1}{t} \int_0^t u(r + h) \, dr = y \quad \text{uniformly for } h \geq 0.$$ 

This implies that $\{u(t)\}$ converges weakly to a fixed point of $\mathcal{S}$ if and only if $\{u(t + h) - u(t)\}$ converges weakly to zero as $t$ tends to infinity for each $h \geq 0$.

1. Introduction

Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a (one-parameter) semigroup acting on $C$, i.e., each $T(t)$ is a self-mapping of $C$ and for each $x$ in $C$ and $t, s \geq 0$ we have

(i) $T(0)x = x$ and
(ii) $T(s + t)x = T(s)T(t)x$.

$\mathcal{S}$ is said to be continuous if for each $x$ in $C$, the mapping $t \to T(t)x$ is continuous. Recall that a semigroup $\mathcal{S}$ on $C$ is said to be

(a) nonexpansive if $\|T(t)x - T(t)y\| \leq \|x - y\|$ for $x, y$ in $C$ and $t \geq 0$;
(b) uniformly asymptotically nonexpansive [6] if there is a function $b : [0, \infty) \to [0, \infty)$ with $\limsup_{t \to \infty} b(t) \leq 1$ such that $\|T(t)x - T(t)y\| \leq b(t)\|x - y\|$ for $x, y$ in $C$ and $t \geq 0$;
(c) asymptotically nonexpansive [6] if for each $x$ in $C$,

$$\limsup_{t \to \infty} \left\{ \sup_{y \in C} [\|T(t)x - T(t)y\| - \|x - y\|] \right\} \leq 0.$$ 

Received by the editors October 20, 1990 and, in revised form, June 4, 1991.
1991 Mathematics Subject Classification. Primary 47H10, 47H20.
Key words and phrases. Almost-orbit, asymptotically nonexpansive semigroup, weakly asymptotically regular, nonexpansive retraction, fixed point, metric projection, asymptotic center.

©1993 American Mathematical Society
0002-9939/93 $1.00 + $.25 per page

385
It is easily seen that (a) ⇒ (b) ⇒ (c) and that both the inclusions are proper (cf. [6, p. 112]).

Recently, Miyadera and Kobayasi [9] introduced the notion of almost-orbits of nonexpansive semigroups on $C$ and proved the weak almost convergence of such an almost-orbit in a uniformly convex Banach space with a Fréchet differentiable norm. Takahashi and Zhang [10] extended this notion to uniformly asymptotically nonexpansive semigroups in Hilbert spaces.

In this paper, we introduce the concept of almost-orbits for asymptotically nonexpansive semigroups. We shall prove that if $u : [0, \infty) \to C$ is an almost-orbit of an asymptotically nonexpansive semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ such that each $T(t)$ is continuous and the fixed point set $F(\mathcal{T})$ of $\mathcal{T}$ is nonempty, then it almost converges weakly to some $y$ in $F(\mathcal{T})$, i.e.,

$$\text{weak-lim}_{t \to \infty} \frac{1}{t} \int_0^t u(r + h) \, dr = y$$

uniformly for $h \geq 0$. This extends Theorem 4.1 of Miyadera and Kobayasi [9] to semigroups of non-Lipschitzian mappings in Hilbert spaces and implies that $\{u(t)\}$ converges weakly to a member of $F(\mathcal{T})$ if and only if $u$ is weakly asymptotically regular, that is, $u(t + h) - u(t) \to 0$ weakly as $t \to \infty$ for each $h \geq 0$. We also prove the existence of a nonexpansive retraction from the set of almost-orbits of the semigroup $\mathcal{T}$ onto the fixed point set $F(\mathcal{T})$ of $\mathcal{T}$. Related results to our paper can be found in Bruck [2], Kirk [5], Lau [7], Takahashi and Zhang [10], and You and Xu [11].

2. Preliminaries

Throughout this section, let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $\mathcal{T} = \{T(t) : t \geq 0\}$ be an asymptotically nonexpansive semigroup acting on $C$. We shall first introduce the notion of almost-orbit of $\mathcal{T}$ and prove some lemmas that will be used later.

**Definition.** A continuous function $u : [0, \infty) \to C$ is said to be an almost-orbit of $\mathcal{T}$ if

$$\lim_{t \to \infty} \left( \sup_{s \geq 0} \|u(s + t) - T(s)u(t)\| \right) = 0.$$

Clearly, for each $x$ in $C$, the orbit $\{T(t)x : t \geq 0\}$ of $\mathcal{T}$ at $x$ is an almost-orbit of $\mathcal{T}$ provided $\mathcal{T}$ is continuous.

Denote by $F(\mathcal{T})$ the fixed point set of $\mathcal{T}$, i.e., $F(\mathcal{T}) = \{x \in C : T(t)x = x \text{ for all } t \geq 0\}$.

**Lemma 2.1.** If each $T(t)$ is continuous then $F(\mathcal{T})$ is closed and convex.

**Proof.** It needs only to prove the convexity of $F(\mathcal{T})$. Suppose $f, g \in F(\mathcal{T})$ and let $p = \frac{1}{2}(f + g)$. For an arbitrary $\varepsilon > 0$, since $\mathcal{T}$ is asymptotically nonexpansive, there exists a $t_0 > 0$ such that for $t \geq t_0$,

$$\sup_{y \in C} \|q - T(t)y\| - \|q - y\| < \varepsilon \quad \text{for } q = f, g,$$

and hence for $t \geq t_0$,

$$\|T(t)p - p\|^2 = \frac{1}{2}\|T(t)p - f\|^2 + \frac{1}{2}\|T(t)p - g\|^2 - \frac{1}{4}\|f - g\|^2 \leq (\frac{1}{2}\|f - g\| + \varepsilon)^2 - \frac{1}{4}\|f - g\|^2.$$
It follows that $\lim_{t \to \infty} \|T(t)p - p\| = 0$, i.e., $T(t)p \to p$ strongly. By continuity of $T(t)$, $p$ is a fixed point of each $T(t)$ and, therefore, $p$ is in $F(F)$. □

**Lemma 2.2.** If $u$ and $v$ are almost-orbits of $F$ then $\lim_{t \to \infty} \|u(t) - v(t)\|$ exists. In particular, for every $f \in F(F)$, $\lim_{t \to \infty} \|u(t) - f\|$ exists.

**Proof.** Set

$$\varphi(t) = \sup_{s \geq 0} \|u(s + t) - T(s)u(t)\|$$

and

$$\psi(t) = \sup_{s \geq 0} \|v(s + t) - T(s)v(t)\|.$$ 

Then $\lim_{t \to \infty} \varphi(t) = \lim_{t \to \infty} \psi(t) = 0$. Since $F$ is asymptotically nonexpansive, for a fixed $t > 0$, one can find a $s_0$ (depending upon $t$) such that for $s \geq s_0$

$$\sup_{y \in C} (\|T(s)u(t) - T(s)y\| - \|u(t) - y\|) < \epsilon;$$

in particular,

$$\|T(s)u(t) - T(s)v(t)\| < \|u(t) - v(t)\| + \epsilon.$$ 

Therefore for $s \geq s_0$,

$$\|u(t + s) - v(t + s)\| \leq \|u(t + s) - T(s)u(t)\| + \|T(s)u(t) - T(s)v(t)\| + \|v(t + s) - T(s)v(t)\| \leq \varphi(t) + \psi(t) + \|u(t) - v(t)\| + \epsilon,$$

and thus for each fixed $t > 0$,

$$\limsup_{s \to \infty} \|u(s) - v(s)\| = \lim_{s \to \infty} \sup_{s \geq 0} \|u(t + s) - v(t + s)\| \leq \varphi(t) + \psi(t) + \|u(t) - v(t)\|.$$ 

It follows that $\limsup_{t \to \infty} \|u(t) - v(t)\| \leq \liminf_{t \to \infty} \|u(t) - v(t)\|$, completing the proof. □

By Lemma 2.1, if $F(F)$ is nonempty, the metric projection $P$ from $H$ onto $F(F)$ is well defined. We now prove

**Lemma 2.3.** Suppose the fixed point set $F(F)$ of $F$ is nonempty. Then for each almost-orbit $u$ of $F$, $\{Pu(t)\}$ converges strongly to some $z$ in $F(F)$. Moreover, if $\{u(t)\}$ converges weakly to a point $y$ in $F(F)$, then $y = z$.

**Proof.** As before, we set $\varphi(t) = \sup_{s \geq 0} \|u(s + t) - T(s)u(t)\|$. First, we show that $\lim_{t \to \infty} \|Pu(t) - u(t)\|$ exists. In fact, for an arbitrary $\epsilon > 0$ and a fixed $t > 0$, we can choose an $s_0$ so large that for $s \geq s_0$

$$\sup_{y \in C} (\|Pu(t) - T(s)y\| - \|Pu(t) - y\|) < \epsilon.$$ 

It then follows that for $s \geq s_0$,

$$\|Pu(t + s) - u(t + s)\| \leq \|Pu(t) - u(t + s)\| \leq \|Pu(t) - T(s)u(t)\| + \|T(s)u(t) - u(s + t)\| \leq \|Pu(t) - u(t)\| + \varphi(t) + \epsilon.$$ 

Since $\epsilon > 0$ is arbitrary, this yields

$$\limsup_{s \to \infty} \|Pu(s) - u(s)\| \leq \liminf_{t \to \infty} \|Pu(t) - u(t)\|.$$
so that $\lim_{t \to \infty} \|Pu(t) - u(t)\|$ exists. Now, since

$$\|Px - y\|^2 \leq \|x - y\|^2 - \|x - Px\|^2$$

for all $x \in H$ and $y \in F(\mathcal{F})$, by setting $x = u(t + s)$ and $y = Pu(t)$, we get for $s \geq s_0$

$$\|Pu(t + s) - Pu(t)\|^2 \leq \|u(t + s) - Pu(t)\|^2 - \|u(t + s) - Pu(t + s)\|^2$$

$$\leq \left(\|Pu(t) - u(t)\| + \varphi(t) + \varepsilon\right)^2 - \|u(t + s) - Pu(t + s)\|^2,$$

which implies that $\limsup_{t \to \infty} (\limsup_{s \to \infty} \|Pu(t) - Pu(s)\|) = 0$, which in turn implies (cf. [1, Lemma 3]) that $\{Pu(t)\}$ converges strongly to some point $z$ in $F(\mathcal{F})$. Now suppose that $\{u(t)\}$ converges weakly to a $y$ in $F(\mathcal{F})$. Then since $\text{Re}(u(t) - Pu(t), f - Pu(t)) \leq 0$ for all $f \in F(\mathcal{F})$, by taking the limit as $t \to \infty$, we get $\text{Re}(y - z, f - z) \leq 0$ for all $f \in F(\mathcal{F})$. Because $y \in F(\mathcal{F})$, we in particular obtain $\|y - z\|^2 = \text{Re}(y - z, y - z) \leq 0$ and thus $y = z$. The proof is complete. $\square$

By applying a result of Lim [8, Corollary 1], the proof of Lemma 5 in Takahashi and Zhang [10] can be modified to prove the following result and is thus omitted.

**Lemma 2.4.** The limit $z$ of $\{Pu(t)\}$ in Lemma 2.3 is the asymptotic center of $\{u(t)\}$ in $F(\mathcal{F})$, i.e., $z$ is the unique point in $F(\mathcal{F})$ that minimizes the functional $\lim_{t \to \infty} \|u(t) - f\|$ over $f$ in $F(\mathcal{F})$.

We remark here that the original notion of asymptotic center was first introduced by Edelstein in [3] to obtain fixed point theorems in [3, 4].

### 3. Asymptotic Behavior

In this section, we prove the asymptotic behavior of an almost-orbit of an asymptotically nonexpansive semigroup in a Hilbert space. The main result, Theorem 3.2, extends Theorem 4.1 of Miyadera and Kobayasi [9] to semigroups of non-Lipschitzian mappings in Hilbert spaces.

**Theorem 3.1.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $\mathcal{S} = \{T(t): t \geq 0\}$ be a continuous asymptotically nonexpansive semigroup acting on $C$ such that each $T(t)$ is continuous. Then the following two statements are equivalent:

(i) $F(\mathcal{S})$ is nonempty;

(ii) there exists a bounded almost-orbit of $\mathcal{S}$.

**Proof.** (i) $\Rightarrow$ (ii). For each $x \in C$, $\{T(t)x: t \geq 0\}$ is an almost-orbit of $\mathcal{S}$. By Lemma 2.2, every almost-orbit of $\mathcal{S}$ is bounded.

(ii) $\Rightarrow$ (i). Let $u$ be a bounded almost-orbit of $\mathcal{S}$. For each $t > 0$, set

$$S_t(u) = \frac{1}{t} \int_0^t u(r) \, dr, \quad \varphi(t) := \sup_{s \geq 0} \|u(s + t) - T(s)u(t)\|.$$

Then $\{S_t(u)\}$ is bounded and $\frac{1}{t} \int_0^t \varphi(r) \, dr \to 0$ as $t \to \infty$. Let $f$ be a weak cluster point of $\{S_t(u)\}$. For an arbitrary $\varepsilon > 0$, we can choose a $t_0$ so large that $\|T(t)u(r) - T(t)f\|^2 \leq \|u(r) - f\|^2 + \varepsilon$ for all $t \geq t_0$ and $r \geq 0$ and that
\( \varphi(t) < \epsilon \) for all \( t \geq t_0 \). Thus for an arbitrary but fixed \( s \geq t_0 \), we have for \( r \geq 0 \) that

\[
-\epsilon \leq \|u(r) - f\|^2 - \|T(s)u(r) - T(s)f\|^2 \\
= \|u(r) - T(s)f\|^2 + \|T(s)f - f\|^2 + 2 \cdot \text{Re}(u(r) - T(s)f, T(s)f - f) \\
- \|T(s)u(r) - u(s + r)\|^2 - \|u(s + r) - T(s)f\|^2 \\
- 2 \cdot \text{Re}(T(s)u(r) - u(s + r), u(s + r) - T(s)f).
\]

Since \( u \) is bounded, \( M = \sup\{|u(t)|: t \geq 0\} < \infty \) and, since \( \{T(s)f: s \geq 0\} \) is an almost-orbit of \( \mathcal{F} \), \( L = \sup\{|T(s)f|: s \geq 0\} < \infty \) by Lemma 2.2. Then it follows for all \( t \geq s \geq t_0 \) that

\[
-\epsilon \leq \frac{1}{t} \int_0^t \|u(r) - T(s)f\|^2 \, dr + \|T(s)f - f\|^2 \\
+ 2 \cdot \text{Re}(S_t(u) - T(s)f, T(s)f - f) - \frac{1}{t} \int_0^t \|u(s + r) - T(s)f\|^2 \, dr \\
- 2 \cdot \text{Re} \left( \frac{1}{t} \int_0^t (T(s)u(r) - u(s + r), u(s + r) - T(s)f) \, dr \right)
\]

\[
\leq \frac{1}{t} \int_0^t \|u(r) - T(s)f\|^2 \, dr - \frac{1}{t} \int_s^{s+t} \|u(r) - T(s)f\|^2 \, dr + \|T(s)f - f\|^2 \\
+ 2 \cdot \text{Re}(S_t(u) - T(s)f, T(s)f - f) + 2(M + L) \frac{1}{t} \int_0^t \varphi(r) \, dr \\
\leq \frac{1}{t} \int_0^s \|u(r) - T(s)f\|^2 \, dr + \|T(s)f - f\|^2 \\
+ 2 \cdot \text{Re}(S_t(u) - T(s)f, T(s)f - f) + 2(M + L) \frac{1}{t} \int_0^t \varphi(r) \, dr,
\]

that is,

\[
-2 \cdot \text{Re}(S_t(u) - T(s)f, T(s)f - f) \leq \frac{1}{t} \int_0^s \|u(r) - T(s)f\|^2 \, dr + \|T(s)f - f\|^2 \\
+ 2(M + L) \frac{1}{t} \int_0^t \varphi(r) \, dr + \epsilon.
\]

Now choose a sequence \( \{t_n\} \), \( t_n \uparrow \infty \), such that \( S_{t_n}(u) \rightharpoonup f \) weakly. Substituting \( t_n \) for \( t \) in the last inequality above and taking the limit as \( n \to \infty \), we arrive at \( \|T(s)f - f\|^2 \leq \epsilon(1 + 2M + 2L) \) for all \( s \geq t_0 \). This shows that \( T(s)f \rightharpoonup f \) in norm and hence \( T(s)f = f \) for all \( t \geq 0 \) by the continuity of the semigroup \( \mathcal{F} \). This completes the proof. \( \square \)

By the same proof as above, we can prove
Corollary 3.1. Let $C$ and $\mathcal{T}$ be as in Theorem 3.1. Suppose $u$ is a bounded almost-orbit of $\mathcal{T}$. Then for every real sequence $\{t_n\}$, $t_n \uparrow \infty$, and any $\{h_n\} \subset [0, \infty)$, every weak limit point of the sequence
\[
\frac{1}{t_n} \int_0^{t_n} u(r + h_n) \, dr, \quad n = 1, 2, \ldots,
\]
is a (common) fixed point of $\mathcal{T}$, i.e., an element in $F(\mathcal{T})$.

We now prove the main result in this paper.

Theorem 3.2. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $\mathcal{T} = \{T(t) : t \geq 0\}$ be a continuous asymptotically nonexpansive semigroup acting on $C$ such that each $T(t)$ is continuous and $F(\mathcal{T})$ is nonempty. Then for every almost-orbit $u$ of $\mathcal{T}$, $\{Pu(t)\}$ converges to a fixed point $z$ of $\mathcal{T}$ and $\{u(t)\}$ almost converges weakly to $z$, i.e.,
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t u(r + h) \, dr = z \quad \text{uniformly for } h \geq 0.
\]

Proof. Since $F(\mathcal{T})$ is nonempty, $\{u(t)\}$ is bounded by Theorem 3.1 and Lemma 2.2 and $\{Pu(t)\}$ converges strongly to some $z$ in $F(\mathcal{T})$ by Lemma 2.3. It remains only to prove that if $\{t_n\} \uparrow \infty$, $\{h_n\} \subset [0, \infty)$, and
\[
\frac{1}{t_n} \int_0^{t_n} u(r + h_n) \, dr \to y
\]
weakly, then $y = z$. To show this, let $M = \sup_{t \geq 0} \|Pu(t) - u(t)\|$. Since for every $r \geq 0$,
\[
\Re(u(r) - Pu(r), Pu(r) - f) \geq 0 \quad \text{for all } f \in F(\mathcal{T}),
\]
we have
\[
\Re(f - z, u(r) - Pu(r)) \leq \Re(Pu(r) - z, u(r) - Pu(r)) \leq M\|Pu(r) - z\| \quad \text{for all } f \in F(\mathcal{T}).
\]

It follows that
\[
\Re(f - z, \frac{1}{t_n} \int_0^{t_n} (u(r + h_n) - Pu(r + h_n)) \, dr) \leq \frac{M}{t_n} \int_0^{t_n} \|Pu(r + h_n) - z\| \, dr.
\]
Since $\lim_{t \to \infty} \|Pu(t) - z\| = 0$, taking the limit as $n$ goes to infinity in the above inequality yields
\[
\Re(f - z, y - z) \leq 0 \quad \text{for all } f \in F(\mathcal{T});
\]
since $y \in F(\mathcal{T})$, we then have $\|y - z\|^2 = \Re(y - z, y - z) \leq 0$ and thus $y = z$. This completes the proof. $\Box$

Theorem 3.3. Under the same assumptions of Theorem 3.2, for each almost-orbit $\{u(t)\}$ of $\mathcal{T}$, the following statements are equivalent:

(i) $\{u(t)\}$ converges weakly to a fixed point of $\mathcal{T}$;

(ii) $\{u(t)\}$ is weakly asymptotically regular, that is, $\{u(t + h) - u(t)\}$ converges weakly to zero as $t$ tends to infinity for each $h \geq 0$;

(iii) $\{u(t+h) - u(t)\}$ converges weakly to zero as $t$ tends to infinity uniformly for $h \geq 0$.
Proof. By Lemma 2.2 and Theorem 3.1, we need only to prove (iii) \(\Rightarrow\) (i). By Theorem 3.2, \(\{u(t)\}\) almost converges weakly to \(z = \lim_{t \to \infty} Pu(t)\). So for each \(w \in H\) and \(\varepsilon > 0\), we can choose a \(t_1\) for which
\[
\left| \langle w, \frac{1}{t_1} \int_0^{t_1} u(r + h) dr - z \rangle \right| < \frac{1}{2} \varepsilon \quad \text{for all } h \geq 0.
\]
Then we choose a \(t_2\) such that
\[
|\langle w, u(t + r) - u(t) \rangle| < \frac{1}{2} \varepsilon \quad \text{for all } t > t_2 \text{ and } 0 \leq r \leq t_1.
\]
Hence for \(t > \max\{t_1, t_2\}\), we have
\[
|\langle w, u(t) - z \rangle| = \left| \langle w, \frac{1}{t_1} \int_0^{t_1} u(t(r) dr - z \rangle \right| \\
\leq \left| \langle w, \frac{1}{t_1} \int_0^{t_1} u(r + t) dr - z \rangle \right| \\
+ \left| \langle w, \frac{1}{t_1} \int_0^{t_1} (u(r + t) - u(t)) dr \rangle \right| \\
\leq \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon = \varepsilon.
\]
and the result follows. \(\square\)

4. Nonexpansive retractions

In this section we prove the existence of a nonexpansive retraction for almost-orbits of an asymptotically nonexpansive semigroup. We begin with

Lemma 4.1. Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\) and \(\mathcal{F} = \{T(t): t \geq 0\}\) be an asymptotically nonexpansive semigroup acting on \(C\). Suppose \(u\) is an almost-orbit of \(\mathcal{F}\). Then for each \(h \geq 0\), the function \(v: [0, \infty) \to C\) defined by \(v(t) = T(h)u(t)\) is also an almost-orbit of \(\mathcal{F}\).

Proof. As before, we set \(\varphi(t) = \sup_{s \geq 0} ||u(s + t) - T(s)u(t)||\). Since
\[
||v(s + t) - T(s)v(t)|| = ||T(h)u(s + t) - T(s)T(h)u(t)|| \\
\leq ||T(h)u(s + t) - u(h + s + t)|| + ||u(h + s + t) - T(s + h)u(t)|| \\
\leq \varphi(s + t) + \varphi(t),
\]
the result follows. \(\square\)

Lemma 4.2. Under the same assumptions of Lemma 4.1, the set
\[
\bigcap_{s \geq 0} \overline{\text{co}}\{u(t): t \geq s\} \cap F(\mathcal{F})
\]
is either empty or the singleton set \(\{z\}\) where \(z = \text{weak-lim}_{t \to \infty}(1/t) \int_0^t u(r) dr = \lim_{t \to \infty} Pu(t)\).

Proof. If \(w\) is in \(\bigcap_{t \geq 0} \overline{\text{co}}\{u(t): t \geq s\} \cap F(\mathcal{F})\) then \(F(\mathcal{F})\) is nonempty, and hence by Theorem 3.2 and Lemma 2.4, \(z := \text{weak-lim}_{t \to \infty}(1/t) \int_0^t u(r) dr = \lim_{t \to \infty} Pu(t)\) is the asymptotic center of \(\{u(t)\}\) in \(F(\mathcal{F})\). We conclude the
proof by showing $w = z$. By Lemma 2.2, $r(w) := \lim_{t \to \infty} \|u(t) - w\|$ and $r := \lim_{t \to \infty} \|u(t) - z\|$ exist and $r(w) \geq r$. Now, since

$$\|z - w\|^2 = \|u(s) - w\|^2 - \|u(s) - z\|^2 - 2 \cdot \Re(z - w, u(s) - z),$$

we get

$$\|z - w\|^2 + 2 \cdot \lim_{s \to \infty} \Re(z - w, u(s) - z)$$

$$= \lim_{s \to \infty} \|u(s) - w\|^2 - \lim_{s \to \infty} \|u(s) - z\|^2 = r(w)^2 - r^2 \geq 0.$$ 

For any $\varepsilon > 0$, there is a $s_0$ so large that for $s \geq s_0$,

$$2 \cdot \Re(z - w, u(s) - z) > -\|z - w\|^2 - \varepsilon.$$

Since $w \in \overline{\{u(t) : t \geq s_0\}}$, it follows that

$$2 \cdot \Re(z - w, w - z) \geq -\|z - w\|^2 - \varepsilon,$$

that is, $\|z - w\|^2 \leq \varepsilon$; since $\varepsilon > 0$ is arbitrary, we must have $z = w$. $\Box$

Theorem 4.1. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\mathcal{S} = \{T(t) : t \geq 0\}$ be a continuous asymptotically nonexpansive semi-group acting on $C$ such that each $T(t)$ is continuous and $F(\mathcal{S})$ is nonempty. Then the limit

$$Q(u) := \text{weak-} \lim_{t \to \infty} \frac{1}{t} \int_0^t u(r) \, dr$$

is the unique retraction from the set $\text{AO}(\mathcal{S})$ of all almost-orbits of $\mathcal{S}$ onto $F(\mathcal{S})$ such that

(i) $Q$ is nonexpansive in the sense that

$$\|Qu - Qv\| \leq \|u - v\|_{\infty} := \sup_{t \geq 0} \|u(t) - v(t)\| \text{ for } u, v \in \text{AO}(\mathcal{S});$$

(ii) $QT(h)u = T(h)Qu = Qu$ for $u \in \text{AO}(\mathcal{S})$ and $h \geq 0$; and

(iii) $Q(u) \in \bigcap_{h \geq 0} \overline{\{u(t) : t \geq s\}}$ for $u \in \text{AO}(\mathcal{S})$.

Proof. (i) By the weak lower semicontinuity of the norm of $H$, we derive that

$$\|Qu - Qv\| \leq \liminf_{t \to \infty} \frac{1}{t} \int_0^t \|u(r) - v(r)\| \, dr \leq \|u - v\|_{\infty};$$

that is, $Q$ is nonexpansive and (i) is proven.

(ii) Let $u \in \text{AO}(\mathcal{S})$ and $h \geq 0$. First we observe that $QT(h)u$ is well defined by Lemma 4.1. Since $Qu \in F(\mathcal{S})$, we have $T(h)Qu = Qu$. Thus it remains to prove $QT(h)u = Qu$. In fact, we have

$$QT(h)u = \text{weak-} \lim_{t \to \infty} \frac{1}{t} \int_0^t T(h)u(r) \, dr$$

$$= \text{weak-} \lim_{t \to \infty} \frac{1}{t} \int_0^t u(h + r) \, dr$$

$$+ \text{weak-} \lim_{t \to \infty} \frac{1}{t} \int_0^t (T(h)u(r) - u(h + r)) \, dr$$

$$= \text{weak-} \lim_{t \to \infty} \frac{1}{t} \int_0^t u(r + h) \, dr = Q(u)$$
since
\[
\frac{1}{t} \int_0^t \| T(h)u(r) - u(h + r) \| \, dr \leq \frac{1}{t} \int_0^t \varphi(r) \, dr \rightarrow 0 \quad \text{as } t \rightarrow \infty
\]
where
\[
\varphi(t) = \sup_{s \geq 0} \| u(s + t) - T(s)u(t) \| \rightarrow 0 \quad \text{as } t \rightarrow \infty.
\]
Finally, (iii) follows directly from Theorem 3.2. The proof is complete. □

REFERENCES