PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 2, February 1993

TOEPLITZ OPERATORS ON CARTAN DOMAINS ESSENTIALLY COMMUTE WITH A BILATERAL SHIFT

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(Communicated by Theodore W. Gamelin)

ABSTRACT. For bounded symmetric domains $\Omega \subset \mathbf{C}^N$, a bilaterial shift operator U is shown to exist on the Bergman space $A^2(\Omega)$ such that $UT_f - T_fU$ is a compact operator for all Toeplitz operators T_f. This may be viewed as an extension of the well-known fact that $S^*TS - T = 0$ whenever T is a Toeplitz operator on H^2, S being the unilateral shift. It also follows that the C^*-algebra generated by Toeplitz operators on $A^2(\Omega)$ does not contain all bounded operators.

Let Ω be a bounded symmetric (Cartan) domain with its standard (Harish-Chandra) realization in $\mathbf{C}^N$ [6], dv the $2n$-dimensional Lebesgue measure on Ω, and $L^2(\Omega, dv)$ the Hilbert space of square-integrable complex-valued functions on Ω. The Bergman space, $A^2(\Omega)$, is the closed subspace of $L^2(\Omega, dv)$ consisting of functions analytic on Ω. Denote by P the orthogonal projection from L^2 onto A^2. For $f \in L^\infty(\Omega)$, the Toeplitz operator $T_f\colon A^2 \to A^2$ and the Hankel operator $H_f\colon A^2 \to L^2 \ominus A^2$ are given by

$$T_f x = P(fx), \qquad H_f x = (I - P)(fx).$$

These operators generalize, in an obvious way, the well-known Toeplitz and Hankel operators on H^2 [5, Chapter 25].

The main result of the present note is the following theorem.

Theorem. *There is a bilateral shift U on $A^2(\Omega)$ such that for all $f \in L^\infty(\Omega)$ the commutator $[U, T_f] \equiv UT_f - T_fU$ is a compact operator.*

Proof. We first introduce some notation and terminology from [1, 2]. Let $\beta(\cdot, \cdot)$ be the Bergman metric on Ω [6] and $\operatorname{dist}(\cdot, \cdot)$ the usual euclidean metric in $\mathbf{C}^N = \mathbf{R}^{2N}$. Denote by $\mathrm{BC}(\Omega)$ the algebra of bounded continuous complex-valued functions on Ω, with $C_\partial(\Omega)$ the subalgebra of functions for which $f(z) \to 0$ as $\operatorname{dist}(z, \partial\Omega) \to 0$. Define

$$\operatorname{Osc}(f, z) = \sup\{|f(w) - f(z)|\colon \beta(z, w) < 1\},$$

the *oscillation* of f at z, and

$$\mathrm{VO}_\partial(\Omega) = \{f \in \mathrm{BC}(\Omega)\colon \operatorname{Osc}(f, \cdot) \in C_\partial(\Omega)\},$$

Received by the editors May 30, 1991.
1991 *Mathematics Subject Classification*. Primary 47B35; Secondary 32M15.
Key words and phrases. Toeplitz operators, bounded symmetric domains.

the algebra of functions with *vanishing oscillation at the boundary.*

Consider the function Φ on Ω given by

$$\Phi(z) = \exp(i\sqrt{\beta(0, z)}).$$

It is well known [1] that $\Phi \in \mathrm{VO}_\partial(\Omega)$. Indeed,

$$\begin{aligned}|\Phi(z) - \Phi(w)| &\le |\beta(0, z)^{1/2} - \beta(0, w)^{1/2}| \\ &\le \frac{|\beta(0, z) - \beta(0, w)|}{\beta(0, z)^{1/2} + \beta(0, w)^{1/2}} \\ &\le \frac{\beta(z, w)}{\beta(0, z)^{1/2} + \beta(0, w)^{1/2}}\end{aligned}$$

and the assertion is immediate since $\beta(0, x) \to +\infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$. Hence, by [2, Theorem B], H_Φ and $H_{\overline{\Phi}}$ are compact operators. According to a well-known commutator identity for Toeplitz operators,

$$T_f T_g - T_g T_f = H_{\overline{g}}^* H_f - H_{\overline{f}}^* H_g$$

for arbitrary $f, g \in L^\infty(\Omega)$. Taking $g = \Phi$, we see that the commutator $T_f T_\Phi - T_\Phi T_f \equiv [T_f, T_\Phi]$ is compact $\forall f \in L^\infty(\Omega)$. Thus the proof will be accomplished if we find a compact operator K such that $T_\Phi + K$ is a bilateral shift with respect to some basis. Since, in particular, $[T_{\overline{\Phi}}, T_\Phi]$ is compact, T_Φ is essentially normal. Hence, by the Brown-Douglas-Fillmore theory [3], it suffices to show that the essential spectrum of T_Φ is the unit circle, $\mathbf{T}$, and that $\operatorname{ind} T_\Phi = 0$.

Denote

$$\mathscr{Q} = \{f \in L^\infty(\Omega) \colon H_f \text{ and } H_{\overline{f}} \text{ are compact}\},$$

and let $\tau(\mathscr{Q})$ be the C^*-algebra generated by $\{T_f \colon f \in \mathscr{Q}\}$. By [1, Theorem B], there is a C^*-isomorphism

$$(*) \qquad \tau(\mathscr{Q})/\text{Compacts} \simeq \mathrm{VO}_\partial / C_\partial(\Omega)$$

which maps T_f into the coset $[f]$ of f in $\mathrm{VO}_\partial / C_\partial$. It follows that, for $f \in \mathscr{Q}$, $\sigma_e(T_f)$ coincides with the spectrum of $[f]$ in $\mathrm{VO}_\partial / C_\partial(\Omega)$. Since $[f]$ is invertible in $\mathrm{VO}_\partial / C_\partial$ iff it is invertible in L^∞ / C_∂, the latter spectrum is easily seen to coincide with the set of all cluster values of f at $\partial\Omega$,

$$\sigma_{\mathrm{VO}_\partial / C_\partial}([f]) = \bigcap_{R>0} \overline{\{f(z) \colon \beta(0, z) > R\}}.$$

Taking $f = \Phi$, we conclude that

$$\sigma_e(T_\Phi) = \bigcap_{R>0} \overline{\{\Phi(z) \colon \beta(0, z) > R\}} = \mathbf{T},$$

which proves the first claim.

To compute the Fredholm index of T_Φ, consider the functions

$$\Phi_m(z) = \exp\left(\frac{i}{m}\sqrt{\beta(0, z)}\right),$$

where m is a positive integer (thus, $\Phi_1 = \Phi$). Everything that was said about Φ is readily seen to apply to Φ_m as well: Φ_m is a bounded continuous function that belongs to VO_∂, so that H_{Φ_m}, $H_{\overline{\Phi_m}}$ are compact operators and $\Phi_m \in \mathscr{Q}$;

further, $\sigma_e(T_{\Phi_m}) = \mathbf{T}$, so T_{Φ_m} are Fredholm operators. Since $\Phi_m^m = \Phi$, it follows from $(*)$ that $T_{\Phi_m}^m = T_\Phi$ modulo the compacts. Thus, we have

$$\operatorname{ind} T_\Phi = \operatorname{ind} T_{\Phi_m}^m = m \cdot \operatorname{ind} T_{\Phi_m}$$

for all positive integers m, which is only possible when $\operatorname{ind} T_\Phi = 0$. The proof is complete.

For Toeplitz operators on the N-dimensional Fock space $A^2(\mathbf{C}^N)$, $N \geq 1$, a similar result was obtained by the present author in [4]. It is also shown there that for arbitrary bounded planar domain $\Omega \subset \mathbf{C}$ there exists a *unilateral* shift operator S on the Bergman space $A^2(\Omega)$ such that, for all Toeplitz operators T_f, $ST_f - T_fS$ is compact. The assertion was also proved for the one-dimensional Fock space $A^2(\mathbf{C})$. These results may be compared with the classical characterization [5, Problem 242] of Toeplitz operators on H^2:

$$T \text{ is a Toeplitz operator} \Leftrightarrow S^*TS = T,$$

S being the unilateral shift on H^2.

As a direct consequence of the above theorem, we have

Corollary. *The C^*-algebra generated by all Toeplitz operators $\{T_f\colon f \in L^\infty(\Omega)\}$, where Ω is a bounded symmetric domain in $\mathbf{C}^N$, does not contain all bounded operators on $A^2(\Omega)$.*

Proof. Let $U = T_\Phi + K$ be the bilateral shift obtained above. Since

$$[A+B, U] = [A, U] + [B, U], \qquad [cA, U] = c[A, U],$$
$$[A^*, U] = U[A, U]^*U, \qquad [AB, U] = A[B, U] + [A, U]B,$$

and

$$\|A_n - A\| \to 0 \quad \text{implies} \quad \|[A_n, U] - [A, U]\| \to 0,$$

we see that the essential commutant, U'^{ess}, of U is a C^*-algebra. According to the preceding theorem, it contains all operators T_f, $f \in L^\infty(\Omega)$. It follows that U'^{ess} contains the C^*-algebra generated by them as well, and we only have to find an operator not belonging to U'^{ess}. Let $\{e_n\}_{n\in\mathbf{Z}}$ be a basis with respect to which U is the bilateral shift, i.e.,

$$Ue_n = e_{n+1}, \qquad n \in \mathbf{Z},$$

and let J be given by

$$Je_n = (-1)^n e_n, \qquad n \in \mathbf{Z}.$$

Then $JU - UJ = 2JU$ is not compact, and so $J \notin U'^{\text{ess}}$.

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