TOEPLITZ OPERATORS ON CARTAN DOMAINS
ESSENTIALLY COMMUTE WITH A BILATERAL SHIFT

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Abstract. For bounded symmetric domains \( \Omega \subset \mathbb{C}^N \), a bilateral shift operator \( U \) is shown to exist on the Bergman space \( A^2(\Omega) \) such that \( UT_f - T_f U \) is a compact operator for all Toeplitz operators \( T_f \). This may be viewed as an extension of the well-known fact that \( S^* T S - T = 0 \) whenever \( T \) is a Toeplitz operator on \( H^2 \), \( S \) being the unipotent shift. It also follows that the \( C^* \)-algebra generated by Toeplitz operators on \( A^2(\Omega) \) does not contain all bounded operators.

Let \( \Omega \) be a bounded symmetric (Cartan) domain with its standard (Harish-Chandra) realization in \( \mathbb{C}^N \), \( dv \) the \( 2n \)-dimensional Lebesgue measure on \( \Omega \), and \( L^2(\Omega, dv) \) the Hilbert space of square-integrable complex-valued functions on \( \Omega \). The Bergman space, \( A^2(\Omega) \), is the closed subspace of \( L^2(\Omega, dv) \) consisting of functions analytic on \( \Omega \). Denote by \( P \) the orthogonal projection from \( L^2 \) onto \( A^2 \). For \( f \in L^\infty(\Omega) \), the Toeplitz operator \( T_f : A^2 \to A^2 \) and the Hankel operator \( H_f : A^2 \to L^2 \otimes A^2 \) are given by

\[
T_f x = P(fx), \quad H_f x = (I - P)(fx).
\]

These operators generalize, in an obvious way, the well-known Toeplitz and Hankel operators on \( H^2 \) [5, Chapter 25].

The main result of the present note is the following theorem.

**Theorem.** There is a bilateral shift \( U \) on \( A^2(\Omega) \) such that for all \( f \in L^\infty(\Omega) \) the commutator \( [U, T_f] \equiv UT_f - T_f U \) is a compact operator.

**Proof.** We first introduce some notation and terminology from [1, 2]. Let \( \beta(\cdot, \cdot) \) be the Bergman metric on \( \Omega \) [6] and \( \text{dist}(-, -) \) the usual euclidean metric in \( \mathbb{C}^N = \mathbb{R}^{2N} \). Denote by \( BC(\Omega) \) the algebra of bounded continuous complex-valued functions on \( \Omega \), with \( C_0(\Omega) \) the subalgebra of functions for which \( f(z) \to 0 \) as \( \text{dist}(z, \partial\Omega) \to 0 \). Define

\[
\text{Osc}(f, z) = \sup\{|f(w) - f(z)| : \beta(z, w) < 1\},
\]

the oscillation of \( f \) at \( z \), and

\[
\text{VO}_0(\Omega) = \{f \in BC(\Omega) : \text{Osc}(f, \cdot) \in C_0(\Omega)\},
\]

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the algebra of functions with vanishing oscillation at the boundary.

Consider the function \( \Phi \) on \( \Omega \) given by
\[
\Phi(z) = \exp(i \sqrt{\beta(0, z)}).
\]
It is well known [1] that \( \Phi \in \text{VO}_\partial(\Omega) \). Indeed,
\[
|\Phi(z) - \Phi(w)| \leq |\beta(0, z)^{1/2} - \beta(0, w)^{1/2}|
\leq \frac{\beta(0, z)^{1/2} + \beta(0, w)^{1/2}}{\beta(0, z)^{1/2} + \beta(0, w)^{1/2}}
\]
and the assertion is immediate since \( \beta(0, x) \to +\infty \) as \( \text{dist}(x, \partial \Omega) \to 0 \).

Hence, by [2, Theorem B], \( H_\Phi \) and \( H_\Phi^* \) are compact operators. According to a well-known commutator identity for Toeplitz operators,
\[
T_fT_g - T_gT_f = H_f^*H_g - H_g^*H_f
\]
for arbitrary \( f, g \in L^\infty(\Omega) \). Taking \( g = \Phi \), we see that the commutator \( T_fT_\Phi - T_\Phi T_f \equiv [T_f, T_\Phi] \) is compact \( \forall f \in L^\infty(\Omega) \). Thus the proof will be accomplished if we find a compact operator \( K \) such that \( T_\Phi + K \) is a bilateral shift with respect to some basis. Since, in particular, \( [T_\Phi, T_\Phi] \) is compact, \( T_\Phi \) is essentially normal. Hence, by the Brown-Douglas-Fillmore theory [3], it suffices to show that the essential spectrum of \( T_\Phi \) is the unit circle, \( \mathbb{T} \), and that \( \text{ind } T_\Phi = 0 \).

Denote
\[
\mathcal{E} = \{ f \in L^\infty(\Omega) : H_f \text{ and } H_f^* \text{ are compact} \},
\]
and let \( \tau(\mathcal{E}) \) be the C*-algebra generated by \( \{ T_f : f \in \mathcal{E} \} \). By [1, Theorem B], there is a C*-isomorphism
\[
\tau(\mathcal{E})/\text{Compacts} \simeq \text{VO}_\partial/\mathcal{C}_\partial(\Omega)
\]
which maps \( T_f \) into the coset \([f]\) of \( f \) in \( \text{VO}_\partial/\mathcal{C}_\partial \). It follows that, for \( f \in \mathcal{E} \), \( \sigma_c(T_f) \) coincides with the spectrum of \([f]\) in \( \text{VO}_\partial/\mathcal{C}_\partial(\Omega) \). Since \([f]\) is invertible in \( \text{VO}_\partial/\mathcal{C}_\partial \) iff it is invertible in \( L^\infty/\mathcal{C}_\partial \), the latter spectrum is easily seen to coincide with the set of all cluster values of \( f \) at \( \partial \Omega \),
\[
\sigma_{\text{VO}_\partial/\mathcal{C}_\partial}([f]) = \bigcap_{R > 0} \{ f(z) : \beta(0, z) > R \}.
\]
Taking \( f = \Phi \), we conclude that
\[
\sigma_{\text{VO}_\partial}([\Phi]) = \bigcap_{R > 0} \{ \Phi(z) : \beta(0, z) > R \} = \mathbb{T},
\]
which proves the first claim.

To compute the Fredholm index of \( T_\Phi \), consider the functions
\[
\Phi_m(z) = \exp\left(\frac{i}{m} \sqrt{\beta(0, z)}\right),
\]
where \( m \) is a positive integer (thus, \( \Phi_1 = \Phi \)). Everything that was said about \( \Phi \) is readily seen to apply to \( \Phi_m \) as well: \( \Phi_m \) is a bounded continuous function that belongs to \( \text{VO}_\partial \), so that \( H_{\Phi_m}, H_{\Phi_m^*} \) are compact operators and \( \Phi_m \in \mathcal{E} \);
further, $\sigma_e(T_{\Phi_m}) = T$, so $T_{\Phi_m}$ are Fredholm operators. Since $\Phi_m' = \Phi$, it follows from (*) that $T_{\Phi_m}' = T_{\Phi}$ modulo the compacts. Thus, we have

$$\text{ind } T_{\Phi} = \text{ind } T_{\Phi_m}' = m \cdot \text{ind } T_{\Phi_m}$$

for all positive integers $m$, which is only possible when $\text{ind } T_{\Phi} = 0$. The proof is complete.

For Toeplitz operators on the $N$-dimensional Fock space $A^2(\mathbb{C}^N)$, $N \geq 1$, a similar result was obtained by the present author in [4]. It is also shown there that for arbitrary bounded planar domain $\Omega \subset \mathbb{C}$ there exists a unilateral shift operator $S$ on the Bergman space $A^2(\Omega)$ such that, for all Toeplitz operators $T_f$, $ST_f - T_fS$ is compact. The assertion was also proved for the one-dimensional Fock space $A^2(\mathbb{C})$. These results may be compared with the classical characterization [5, Problem 242] of Toeplitz operators on $H^2$:

$$T \text{ is a Toeplitz operator } \iff S^*TS = T,$$

$S$ being the unilateral shift on $H^2$.

As a direct consequence of the above theorem, we have

**Corollary.** The $C^*$-algebra generated by all Toeplitz operators $\{T_f: f \in L^\infty(\Omega)\}$, where $\Omega$ is a bounded symmetric domain in $\mathbb{C}^N$, does not contain all bounded operators on $A^2(\Omega)$.

**Proof.** Let $U = T_{\Phi} + K$ be the bilateral shift obtained above. Since

$$[A + B, U] = [A, U] + [B, U], \quad [cA, U] = c[A, U],$$


and

$$\|A_n - A\| \to 0 \implies \|[A_n, U] - [A, U]\| \to 0,$$

we see that the essential commutant, $U^\text{ess}$, of $U$ is a $C^*$-algebra. According to the preceding theorem, it contains all operators $T_f, f \in L^\infty(\Omega)$. It follows that $U^\text{ess}$ contains the $C^*$-algebra generated by them as well, and we only have to find an operator not belonging to $U^\text{ess}$. Let $\{e_n\}_{n \in \mathbb{Z}}$ be a basis with respect to which $U$ is the bilateral shift, i.e.,

$$Ue_n = e_{n+1}, \quad n \in \mathbb{Z},$$

and let $J$ be given by

$$Je_n = (-1)^ne_n, \quad n \in \mathbb{Z}.$$

Then $JU - UJ = 2JU$ is not compact, and so $J \notin U^\text{ess}$.

**References**


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