DENSE PURE SUBGROUPS OF LOCALLY COMPACT GROUPS

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Abstract. We prove that a locally compact abelian (LCA) group has no proper dense pure subgroups if and only if it does not have proper dense subgroups. This solves a problem of Armacost.

The following problem was posed by Dietrich in [2]: does every nondiscrete LCA group have a dense proper subgroup? Rajagopalan and Subrahmanian [7] gave a negative answer to this question. Later, Khan [5] and the author [4] proved that an LCA group \( G \) has no proper dense subgroups if and only if \( t(G) \) and \( pG \) are open in \( G \) for each prime \( p \), where \( t(G) \) is the maximal torsion subgroup in \( G \) and \( pG = \{px \mid x \in G\} \). Among numerous questions in the book of Armacost [1] we mention the following two:

(i) Is it true that an LCA group \( G \) is discrete whenever every pure subgroup of \( G \) is closed? (Question 7.24).
(ii) What are the LCA groups without proper dense pure subgroups? (Question 7.18).

We recall that a subgroup \( H \) of an abelian group \( G \) is said to be pure in \( G \) if \( nH = nG \cap H \) for all integers \( n \). Takahashi [8] gave a positive answer to the first question and a partial answer to the second. He proved that a nondiscrete LCA group \( G \) contains \( 2^c \) proper dense pure subgroups whenever \( G \) is compact or \( t(G) \) is not open in \( G \). We shall prove that the following holds.

Theorem. For a nondiscrete LCA group \( G \), the following conditions are equivalent:

(i) the subgroups \( t(G) \) and \( pG \) are open for all prime \( p \);
(ii) \( G \) has no proper dense subgroups;
(iii) \( G \) has no proper dense pure subgroups.

We begin with three lemmas.

Lemma 1. Let \( G \) be an LCA group, \( p \) a prime. If the subgroup \( pG \) is not open in \( G \), then for each \( a \in G \setminus pG \) there exists a dense subgroup \( G' \) of index \( p \) in \( G \) such that \( a \notin G' \).

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Proof. If there is a subgroup $F$ in $G$ with $pG \subset F$ and $a \in \overline{F \setminus F}$, then any maximal subgroup among those containing $F$ and avoiding $a$ is the required subgroup $G'$. (Here and in what follows, $\overline{X}$ is the closure of a subset $X$ in $G$.) To construct $F$ we first choose a nonclosed subgroup $H$ in $G$ containing $pG$ with $a \notin H$. If $a \in \overline{H}$ then $F = H$. Otherwise, $F = \text{gr}(a+b, H)$ where $b \in H \setminus H$. Indeed, since $H \subset F$, it follows that $b \in F$, and hence $a \in F$. On the other hand, if $a \in F$ then $a = k(a+b)+h$, where $k$ is an integer, $h \in H$; i.e., $(1-k)a = kb + h \in \overline{H}$. Since $pG \subset H$ and $a \notin H$, we have $1-k \equiv 0 \pmod{p}$, therefore, $kb + h \equiv b + h \equiv 0 \pmod{pG}$, i.e., $b \in H$—a contradiction. Thus $a \in \overline{F \setminus F}$ and this completes the proof.

Lemma 2. If $G'$ is a dense subgroup of prime index in an LCA group $G$ and $t(G) \subset G'$, then there exists a subgroup $G''$ of $G$ such that

(i) $G'' \subset G'$;

(ii) $G''$ is dense in $G$;

(iii) $G = G'' \oplus A$, where $A$ is a finite cyclic group.

Proof. Let $p$ be the index of $G'$ in $G$, and $a$ an element of the smallest order among those elements of $t(G)$ not belonging to $G'$, $p'$ being the order of $a$, $r \geq 1$. Let $b = pa$, $A = \text{gr}(a)$, $B = \text{gr}(b)$. We shall show that $B$ is a pure subgroup of $G'$. It suffices to check the equality $p'^{-1}G' \cap B = 0$ [3, Proposition 27.1]. Suppose the contrary, $p'^{-1}x = nb$, $n = p^s m$, $0 \leq s < r - 1$, $m$ is not divided by $p$, $x \in G'$. Then $r - s - 2 \geq 0$, the element $a' = p^{r-s-2}x - ma$ does not belong to $G'$, and

$$p^{s+1}a' = p'^{-1}x - nb = 0.$$  

Since $s + 1 < r$, the order of $a'$ is less than that of $a$, contradicting the choice of $a$.

Since $B$ is pure in $G'$, $G' = G'' \oplus B$ for a subgroup $G''$ of $G'$. From this we deduce that $G = G'' \oplus A$. If $G''$ is not dense in $G$, then $G'' = G'' \oplus A'$, where $A'$ is a proper subgroup of $A$. Hence, $A' \subset B$, whence $G'' \subset G'$. Since $G''$ is a closed subgroup of finite index in $G$, it follows that $G''$ is open, and hence $G'$ is clopen. This contradicts the condition. Thus $G''$ is dense in $G$ and the proof is complete.

The following lemma is contained in [6, Lemmas 4 and 5]. We provide a slightly different proof.

Lemma 3. If the subgroup $t(G)$ of torsion elements in an LCA group $G$ is not open, then $G$ has a dense subgroup $M$ such that $G/M$ is isomorphic to the additive group $\mathbb{Q}$ of rationals.

Proof. First we note that any compact nontorsion group $H$ has a free abelian subgroup of infinite (continuum) rank. This is true if the connected component $H$ of the identity is not trivial. If $H$ is totally disconnected then $H$ contains infinite closed monothetic subgroups for which this fact follows from [1, 5.5(e)].

Let $F$ be a free abelian group of infinite index that is contained in an open compact subgroup $H$ of $G$. Let $D$ be a subgroup of $F$ with $F/D \simeq \mathbb{Q}$ (\simeq means isomorphism). Then

$$G/D = F/D \oplus M/D$$
for a subgroup $M$ of $G$. It follows from (1) that $G/M \cong \mathbb{Q}$. Let $L = H \cap M$. It then follows from (1) that

$$H = F + L \quad \text{and} \quad H/L \cong F/F \cap L = F/D \cong \mathbb{Q}.$$

Hence, $L$ is dense in $H$, i.e., $H \subset \overline{M}$. Since $F \subset H$, it follows that $F \subset \overline{M}$. Now it follows from (1) that $\overline{M} = G$. The proof is complete.

We proceed to the proof of the theorem. The only implication of this theorem that we have to prove is (iii) $\Rightarrow$ (i). By Lemma 3 $t(G)$ is open in $G$, and by Lemmas 1 and 2 the subgroup $pG$ is open for each prime $p$.

References


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