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STRONG TYPE ENDPOINT BOUNDS FOR ANALYTIC FAMILIES OF FRACTIONAL INTEGRALS

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ABSTRACT. In $\mathbb{R}^2$ we consider an analytic family of fractional integrals, whose convolution kernel is obtained by taking some transverse derivatives of arclength measure on the parabola (t, t^2) multiplied by $|t|^\gamma$ and doing so in a homogeneous way. We determine the exact range of p, q for which the analytic family maps L^p to L^q. We also resolve a similar issue on the Heisenberg group.

1. INTRODUCTION

In $\mathbb{R}^2$ consider the following family of operators:

$$S^\gamma(f)(x) = \int_{-\infty}^{\infty} f(x_1 - t, x_2 - t^2)|t|^\gamma \frac{dt}{t} \quad \text{where } 0 \le \gamma \le 1, \tag{1.1}$$

where the integral in (1.1) is interpreted in the principal value sense when $\gamma = 0$. For $\gamma > 0$, the operators S^γ are called fractional integrals along the parabola (t, t^2) and have been studied by Ricci and Stein [RS] and Christ [C2], who determined the range of $(1/p, 1/q, \gamma)$ for which S^γ maps L^p to L^q. By homogeneity such a boundedness result can happen only when $1/p - 1/q = \operatorname{Re}\gamma/3$. In $\mathbb{R}^2$, let Δ be the closed triangle with vertices $(0, 0)$, $(1, 1)$, and $(2/3, 1/3)$, and let Γ be the part of Δ that does not contain the diagonal. [RS] proved $L^p \to L^q$ boundedness for S^γ when $(1/p, 1/q)$ lie in Γ minus the piece of the boundary $\{(1/p, 1/q): q = 2p \text{ and } 2 < p < \infty\}$ union its reflection across the line $1/q = 1 - 1/p$. [C2] proved $L^p \to L^q$ boundedness for the remaining boundary points of Γ that do not lie on the diagonal $p = q$. (When $\gamma = 0$, S^0 is the Hilbert transform along the parabola and it is bounded on the diagonal for $1 < p < \infty$. See [SWA] for details.) Furthermore it is known from [RS] that no positive result for S^γ holds outside Γ when $\gamma > 0$.

We prove a similar result as in [C2, RS], for an analytic family of fractional integrals along the parabola S_z^γ, in which the operators S^γ can be embedded. The convolution kernel of S_z^γ is obtained by taking $-z-1$ transverse derivatives of arclength measure on the parabola, multiplied by $|t|^\gamma$, and doing so in a homogeneous way. The analytic family S_z^γ is defined in such a way as to satisfy $S_{-1}^\gamma = S^\gamma$.

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We now give a precise definition of S_z^γ. Fix an even nonnegative function $\psi \in C_0^\infty(\mathbb{R})$ supported in $[-1, 1]$ and equal to 1 on $[-1/2, 1/2]$. Also fix $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma \geq 0$. For f smooth with compact support in $\mathbb{R}^2$, we define

$$(S_z^\gamma f)(x) = \int_{-\infty}^{\infty} \int 2\Gamma\left(\frac{z+1}{2}\right)^{-1} |u-1|^z \psi(u-1) f(x_1 - t, x_2 - ut^2)\,du\,|t|^\gamma \frac{dt}{t},$$

where the outer integral is to be interpreted in the principal value sense, when $\operatorname{Re}\gamma = 0$. S_z^γ is initially defined for $\operatorname{Re} z > -1$. By analytic continuation, see [GS], the definition of S_z^γ can be extended for all z complex. Because of the Γ function normalization we get that $S_{-1}^\gamma = S^\gamma$, for all γ with $\operatorname{Re}\gamma \geq 0$. S_z^γ depends analytically on both γ and z and, therefore, is a double analytic family of operators with parameters $(z, \gamma) \in \mathbb{C} \times \mathbb{C}_+$, where by $\mathbb{C}_+$ we denote the set of all complex numbers with nonnegative real part.

Our first result describes the exact range of $(1/p, 1/q, z, \gamma)$ for which S_z^γ maps $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ when $\gamma > 0$. Since such a boundedness result can only hold when $\operatorname{Re}\gamma/3 = 1/p - 1/q$, it is enough to describe the possible range of p, q, and z. Our first theorem is the following:

Theorem 1. *For* $\operatorname{Re}\gamma > 0$, *the analytic family of fractional integrals* S_z^γ *maps* L^p *to* L^q *if and only if* $(1/p, 1/q, \operatorname{Re} z)$ *lies* on or vertically above *the interiors of the faces* BCD *and* ABD *union the edge* BD $-$ {B} *of the tetrahedron* ABCD *with vertices* A $= (0, 0, -1)$, B $= (1/2, 1/2, -3/2)$, C $= (1, 1, -1)$, *and* D $= (1, 0, 0)$. (*See Figure* 1.)

We use Theorem 1 in [C2] to treat the main part of the kernel of S_z^γ for a certain range of z's but we do not follow the method of Christ's proof since the positivity of the kernel of S^γ was essential in the treatment of this operator in his work. Throughout this paper $C_{z,\gamma}$, $c_{z,\gamma}$ will denote constants that grow

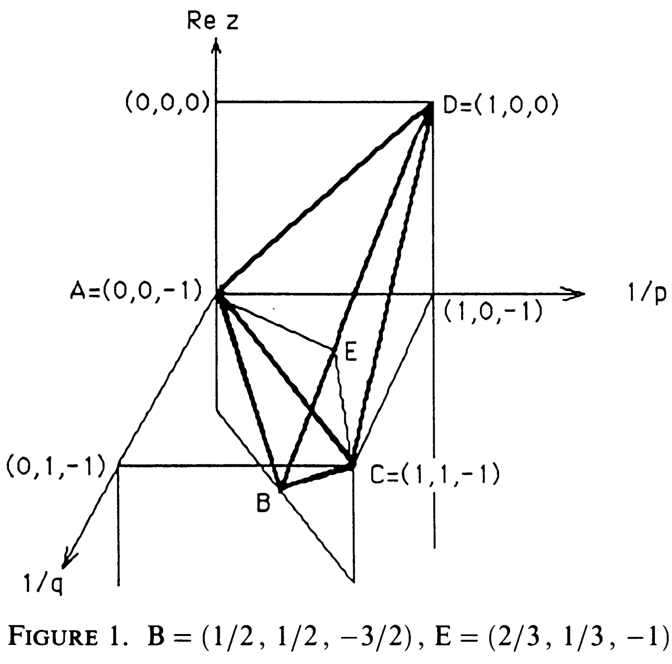

FIGURE 1. B $= (1/2, 1/2, -3/2)$, E $= (2/3, 1/3, -1)$

at most exponentially as $|\operatorname{Im}\gamma|$, $|\operatorname{Im} z| \to +\infty$. These constants will be called of admissible growth.

2. The easy estimates

In this section we prove the endpoint estimates corresponding to the vertices $(1/2, 1/2, -3/2)$ and $(1, 0, 0)$ of the tetrahedron. More precisely, we have the following

Proposition. (1) *S_z^γ maps $L^2 \to L^2$ when $\operatorname{Re} z = -3/2$ and $\operatorname{Re}\gamma = 0$.*
(2) *S_z^γ maps $L^1 \to L^\infty$ when $\operatorname{Re} z \geq 0$ and $\operatorname{Re}\gamma = 3$.*
In both cases the bounds are of admissible growth in $|\operatorname{Im}\gamma|$, $|\operatorname{Im} z| \to +\infty$.

Proof. We start by proving (2), the easier of the two estimates. Fix γ with $\operatorname{Re}\gamma = 3$ and z with $\operatorname{Re} z = 0$. We have

$$\begin{aligned}
&|(S_z^\gamma f)(x)| \\
&\leq 2\left|\Gamma\left(\frac{z+1}{2}\right)^{-1}\right| \iint |u-1|^{\operatorname{Re} z}\psi(u-1)|f(x_1-t, x_2-ut^2)|du|t|^{\operatorname{Re}\gamma}\frac{dt}{|t|} \\
&\leq C_z \iint \left|\frac{w}{t^2}-1\right|^{\operatorname{Re} z}\psi\left(\frac{w}{t^2}-1\right)|f(x_1-t, x_2-w)|\,dw\,dt \leq C_z\|f\|_{L^1}
\end{aligned}$$

and this proves (2).

We continue with the proof of (1). Fix $z = -3/2 + i\theta$ and $\gamma = i\rho$ until the end of this section. Denote by D_z the distribution:

$$\langle D_z, f\rangle = \int f(u)2\Gamma\left(\frac{z+1}{2}\right)^{-1}|u-1|^z\psi(u-1)\,du.$$

(Again D_z is originally defined for $\operatorname{Re} z > -1$ and is analytically continued for all z complex.) Let us call K_z^γ the convolution kernel of S_z^γ. Direct calculation shows that

$$\widehat{K_z^\gamma}(\xi_1, \xi_2) = \lim_{\substack{\varepsilon\to 0\\ N\to+\infty}} \int_{|t|=\varepsilon}^{N} \widehat{D}_z(t^2\xi_2)e^{-2\pi i t\xi_1}|t|^{i\rho}\frac{dt}{t}.$$

(The limits are easily shown to exist.) We have that

$$\begin{aligned}
\widehat{D_z}(v) &= 2\left(\Gamma\left(\frac{z+1}{2}\right)^{-1}|u|^z\psi(u)\right)^{\wedge}(v)e^{-2\pi i v} \\
&= c2^z\Gamma\left(-\frac{z}{2}\right)^{-1}(|\cdot|^{-z-1} * \hat{\psi})(v)e^{-2\pi i v}, \qquad c \neq 0,
\end{aligned}$$

where in the last equality we used a formula on page 359 in [GS]. The behavior of $L_z(v) = (|\cdot|^{-z-1} * \hat{\psi})(v)$ at ∞ will be of importance in the study of the Fourier transform of K_z^γ. It is easy to see that L_z is an even C^∞ function on the real line and, by Lemma 3.2 in [G1], we have that

$$L_z(v) = c_z|v|^{-z-1} + O(|v|^{-M}) \quad \forall M > 0 \text{ as } |v| \to \infty,$$

where all the constants above are of admissible growth and c_z is nonzero. We will prove that $\widehat{K_z^\gamma}(\xi_1, \xi_2)$ is bounded. Fix $\xi_2 \neq 0$, and let $\varepsilon' = \varepsilon|\xi_2|^{-1/2}$,

$N' = N|\xi_2|^{-1/2}$, $\lambda = \xi_1|\xi_2|^{-1/2}$, and $\varepsilon_2 = \operatorname{sgn}\xi_2$. Also let $a = c_{z,\gamma}$ be a positive large constant to be chosen later. By the evenness of L_z we get

$$\widehat{K_z^\gamma}(\xi_1, \xi_2) = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{|t|=\varepsilon}^{N} C_z L_z(t^2\xi_2) e^{-2\pi i(t\xi_1+\xi_2 t^2)}|t|^{i\rho}\frac{dt}{t}$$
$$= \lim_{\substack{\varepsilon'\to 0\\ N'\to\infty}} \int_{|t|=\varepsilon'}^{N'} C_z|\xi_2|^{-i\rho/2} L_z(t^2) e^{-2\pi i(t\lambda+\varepsilon_2 t^2)}|t|^{i\rho}\frac{dt}{t}.$$

We now write $\widehat{K_z^\gamma}(\xi_1, \xi_2)$ as the sum of

$$\lim_{\varepsilon'\to 0} \int_{|t|=\varepsilon'}^{a} C_z|\xi_2|^{-i\rho/2} L_z(t^2) e^{-2\pi i(t\lambda+\varepsilon_2 t^2)}|t|^{i\rho}\frac{dt}{t}, \tag{2.1}$$

$$\lim_{N'\to\infty} \int_{|t|=a}^{N'} C_z|\xi_2|^{-i\rho/2} L_z(t^2) e^{-2\pi i(t\lambda+\varepsilon_2 t^2)}|t|^{i\rho}\frac{dt}{t}. \tag{2.2}$$

Because of the smoothness of L_z at 0, (2.1) remains always bounded by a constant of admissible growth for all λ real. By the asymptotic expansion of L_z at ∞, we have that

$$(2.2) = \lim_{N'\to\infty} \int_{|t|=a}^{N'} C_z|\xi_2|^{-i\rho/2}|t^2|^{-z-1} e^{-2\pi i(t\lambda+\varepsilon_2 t^2)}|t|^{i\rho}\frac{dt}{t}$$
$$+ \text{ a remainder term that is bounded uniformly in } \lambda.$$

The main term above is equal to

$$C_z|\xi_2|^{-i\rho/2} \lim_{N'\to\infty} \int_{t=a}^{N'} t^{i(\rho-2\theta)} e^{-2\pi i\varepsilon_2 t^2}(e^{-2\pi i t\lambda} - e^{2\pi i t\lambda})\, dt. \tag{2.3}$$

The phase function $\phi(t) = -2\pi i\varepsilon_2(t^2 \pm t\lambda) + i(\rho - 2\theta)\ln t$, has second derivative ϕ'' that satisfies $|\phi''(t)| \geq c_{z,\gamma}$ if $t \geq a$ and a is large enough. Van der Corput's Lemma [Z, p. 197] now gives that the integral in (2.3) is bounded by a constant uniformly in N' and λ. Therefore $\widehat{K_z^\gamma}$ is bounded and our proposition is now proved.

3. The main estimates

So far, we have proved the estimates corresponding to the vertices $(1/2, 1/2, -3/2)$ and $(1, 0, 0)$ of the tetrahedron. By interpolation, we get estimates for the edge in between. No strong type estimates are true for the remaining vertices, for it is known that $S_{-1}^0 = S^0$ does not map $L^1 \to L^1$ nor $L^\infty \to L^\infty$. Our next goal is to fill in the sides. The main result of this section is

Proposition. *For* $\operatorname{Re} z = -1$ *and* $\operatorname{Re}\gamma = 3/2p$, S_z^γ *maps* L^p *to* L^{2p} *with bounds of admissible growth, whenever* $3/2 \leq p < \infty$.

Proof. On the real line call h_z the distribution $h_z(u) = 2\Gamma(\frac{z+1}{2})^{-1}|u|^z\psi(u)$, originally defined for $\operatorname{Re} z > -1$ and extended for all z by analytic continuation. Let H_z be the distribution on $\mathbb{R}^2$ defined by $\delta_{x_1=0}h_z(x_2)$. By $\mu_{z,\gamma}$ we will denote the measure acting on functions f as

$$\langle \mu_{z,\gamma}, f\rangle = \int_{|t|\leq 1} f(t, t^2)|t|^{\gamma-2z-2}\frac{dt}{t}.$$

Fix p, z, and γ as in the statement of the theorem. Let $q = 2p$. The basic property of $\mu_{z,\gamma}$ is that it convolves L^p to L^q. This is because of Theorem 1 in [C2] that justifies the third inequality:

$$\begin{aligned}\|\mu_{z,\gamma} * f\|_{L^q} &\leq \left\| \int_{|t|\leq 1} |f(x_1 - t, x_2 - t^2)|\, |t|^{\operatorname{Re}\gamma - 1}\, dt \right\|_{L^q} \\ &\leq \left\| \int_{\mathbf{R}} |f(x_1 - t, x_2 - t^2)|\, |t|^{\operatorname{Re}\gamma - 1} dt \right\|_{L^q} \leq C_{p,\gamma} \|f\|_{L^p}.\end{aligned}$$

We now continue the proof of our theorem. We need to prove that

$$\left\| \int \langle D_z(u), f(x_1 - t, x_2 - ut^2)\rangle |t|^\gamma \frac{dt}{t} \right\|_{L^q} \leq C_{p,z,\gamma} \|f\|_{L^p}.$$

It suffices to prove that

$$\left\| \int_{|t|\leq M} \langle D_z(u), f(x_1 - t, x_2 - ut^2)\rangle |t|^\gamma \frac{dt}{t} \right\|_{L^q} \leq C_{p,z,\gamma} \|f\|_{L^p} \tag{3.1}$$

is valid for all $M > 0$ with a bound $C_{p,z,\gamma}$ independent of $M > 0$. To prove (3.1), by homogeneity we may assume that $M = 1$. Let $K^\gamma_{z,1}$ be the convolution kernel of the operator in (3.1) when $M = 1$.

By χ_A we denote the characteristic function of the set A. We have the following

Lemma. $K^\gamma_{z,1} = H_z * \mu_{z,\gamma} + \zeta(x)$ *where* $\zeta(x)$ *satisfies*

$$|\zeta(x)| \leq C_{z,\gamma} \left| \frac{x_2}{x_1^2} - 1 \right|^{-1} \chi_{|x_2/x_1^2 - 1| \geq \frac{1}{2}} |x_1|^{\operatorname{Re}\gamma - 3}.$$

Proof. Let $\tilde{\mu}_{z,\gamma}$ denote the reflection of the measure $\mu_{z,\gamma}$ about the origin. For all Schwartz functions g we have

$$\begin{aligned}\langle H_z * \mu_{z,\gamma}, g\rangle &= \langle H_z, \tilde{\mu}_{z,\gamma} * g\rangle \\ &= \left\langle H_z(x_1, x_2), \int_{|t|\leq 1} g(x_1 + t, x_2 + t^2)|t|^{\gamma - 2z - 2} \frac{dt}{t} \right\rangle \\ &= \int_{|t|\leq 1} 2\Gamma\left(\frac{z+1}{2}\right)^{-1} |x_2|^z \psi(x_2) \int g(t, x_2 + t^2)|t|^{\gamma - 2z - 2} \frac{dt}{t}\, dx_2 \\ &= \int_{|x_1|\leq 1} \int 2\Gamma\left(\frac{z+1}{2}\right)^{-1} |x_2 - x_1^2|^z \psi(x_2 - x_1^2) g(x_1, x_2) \\ &\qquad\qquad \times |x_1|^{\gamma - 2z - 2} x_1^{-1}\, dx_1\, dx_2.\end{aligned}$$

It follows that $\langle K^\gamma_z - H_z * \mu_{z,\gamma}, g\rangle = \iint \zeta(x_1, x_2) g(x_1, x_2)\, dx_1\, dx_2$, where

$$\zeta(x_1, x_2) = 2\Gamma\left(\frac{z+1}{2}\right)^{-1} \left| \frac{x_2}{x_1^2} - 1 \right|^z \frac{|x_2|^{\gamma - 2}}{x_1} \left[\psi\left(\frac{x_2}{x_1^2} - 1\right) - \psi(x_2 - x_1^2) \right]_{\chi_{|x_1|\leq 1}}.$$

Clearly $\zeta(x_1, x_2)$ satisfies the asserted estimate and this concludes the proof of the lemma.

Note that $\widehat{H}_z(\xi_1, \xi_2) = \hat{h}_z(\xi_2) = c_z(|\xi_2|^{-z-1} * \hat{\psi}(\xi_2))$. Since $\operatorname{Re} z = -1$, the Hörmander multiplier theorem [S2, pp. 51–52] gives that convolution with h_z

is a bounded operator on $L^p(\mathbb{R})$ for $1 < p < \infty$ and, therefore, convolution with H_z is a bounded operator on $L^p(\mathbb{R}^2)$ for the same range of p's. Thus

$$\|f * H_z * \mu_{z,\gamma}\|_{L^q} \le C_{p,z,\gamma}\|f * H_z\|_{L^p} \le C_{p,z,\gamma}\|f\|_{L^p}.$$

It remains to control $\|f * \zeta\|_{L^q}$ by $C_{p,\gamma}\|f\|_{L^p}$. We prove that $\zeta \in L^{r,\infty}$ where $r = 3/(3 - \operatorname{Re}\gamma)$. We denote by $|A|$ the Lebesgue measure of the set A. Let α be a positive number and set $\beta = \alpha^{-1/(\operatorname{Re}\gamma - 3)}$. Computation gives

$$\begin{aligned}
|\{x : |\zeta(x)| > \alpha\}| &\le \left|\left\{x : \left|\frac{x_2}{x_1^2} - 1\right|^{-1} \chi_{|x_2/x_1^2 - 1| \ge 1/2} |x_1|^{\operatorname{Re}\gamma - 3} > \alpha\right\}\right| \\
&= \left|\left\{x : \left|\frac{\beta^2 x_2}{(\beta x_1)^2} - 1\right|^{-1} \chi_{|\beta^2 x_2/(\beta x_1)^2 - 1| \ge 1/2} |\beta x_1|^{\operatorname{Re}\gamma - 3} > 1\right\}\right| \\
&= \beta^{-3}\left|\left\{(x_1, x_2) : \left|\frac{x_2}{x_1^2} - 1\right|^{-1} \chi_{|x_2/x_1^2 - 1| \ge 1/2} |x_1|^{\operatorname{Re}\gamma - 3} > 1\right\}\right| \\
&= \alpha^{-r} m,
\end{aligned}$$

where

$$m = \left|\left\{(x_1, x_2) : \left|\frac{x_2}{x_1^2} - 1\right|^{-1} \chi_{|x_2/x_1^2 - 1| \ge 1/2} |x_1|^{\operatorname{Re}\gamma - 3} > 1\right\}\right|.$$

We next show that $m < \infty$. This amounts to showing that the total area bounded by the following equations in $\mathbb{R}^2$ is finite,

$$\tfrac{3}{2}x_1^2 \le x_2 \le x_1^2 + |x_1|^{\operatorname{Re}\gamma - 1}, \qquad x_1^2 - |x_1|^{\operatorname{Re}\gamma - 1} \le x_2 \le \tfrac{1}{2}x_1^2.$$

This last assertion is obvious and is due to the fact that $0 < \operatorname{Re}\gamma \le 1$. We have now proved that $\zeta \in L^{r,\infty}$ where $r = 3/(3 - \operatorname{Re}\gamma)$. It follows from Young's inequality that convolution with ζ maps L^p to L^q, where p, q, and r are related as in $1/r + 1/p = 1 + 1/q$, which is equivalent to

$$\frac{3 - \operatorname{Re}\gamma}{3} + \frac{1}{p} = 1 + \frac{1}{q} \quad \text{or} \quad \operatorname{Re}\gamma = \frac{3}{2p}.$$

This concludes the proof of the main result of this section.

4. Conclusion of the proof of Theorem 1

We use the estimates of the previous sections and analytic interpolation to prove Theorem 1. We also show that this theorem describes the exact range of p, q, z, and γ such that S_z^γ maps L^p to L^q, when $\gamma > 0$. Recall that $\mathrm{A} = (0, 0, -1)$, $\mathrm{B} = (1/2, 1/2, -3/2)$, $\mathrm{C} = (1, 1, -1)$, and $\mathrm{D} = (1, 0, 0)$ are the vertices of the tetrahedron, and let E be the point $(2/3, 1/3, -1)$. (See Figure 1.) As we mentioned before, interpolation between the points B and D gives that on the edge BD our analytic family maps L^p to $L^{p'}$. (See Proposition in §2.) By the proposition in §3, we have strong type bounds on the closed segment EC minus the point C. We now interpolate between the edges BD and $\mathrm{BE} - \{\mathrm{C}\}$ to get strong type bounds on the interior of the face BCD. By duality we also fill in the interior of the face ABD. When $\gamma > 0$, we have now proved strong type bounds on the interior of the bottom faces of the critical tetrahedron ABCD union the point D. Finally by interpolation we get strong type bounds for every point that lies vertically above.

The best result known on the line segment BC is that S_z^0 maps L^p to $L^{p,p'}$, see [G1]. By duality we get that on the line segment AC, S_z^0 maps $L^{p',p}$ to $L^{p'}$. It is easy to check that no strong type bounds hold on the open segments CD and AD. However, using the fact that the analytic family S_z^γ maps the space parabolic H^1 to weak L^1 when $\operatorname{Re}\gamma = 0$ and $\operatorname{Re} z = -1$ [G1, Theorem 2], interpolation gives that on the open line segment CD, S_z^γ maps H^1 to weak L^p. Finally by duality we get that on the open line segment AD, S_z^γ maps $L^{p',1}$ to parabolic BMO.

We now indicate why no boundedness results hold below the faces BCD and ABD of tetrahedron. Let $\delta > 0$ be small and let f_δ be the characteristic function of the square of sidelength δ centered at the origin. Since away from the parabola the kernel K_z^γ looks like

$$K_z^\gamma(x) = c_z|x_1|^{-2-2z+\gamma}x_1^{-1}|x_2 - x_1^2|^z\psi(x_2/x_1^2 - 1),$$

it follows that on the set $A_\delta = \{x : x_1 \sim 1 \text{ and } |x_2 - x_1^2| \geq 10\delta\}$, $|(S_z^\gamma f_\delta)|$ looks like

$$|(S_z^\gamma f_\delta)(x)| \sim |x_2 - x_1^2|^z\delta^2.$$

Therefore,

$$\left(\int_{A_\delta} |(S_z^\gamma f_\delta)(x)|^q dx\right)^{1/q} \sim \delta^2\delta^{\operatorname{Re} z + 1/q},$$

and since $\|f_\delta\|_{L^p} = \delta^{2/p}$, letting $\delta \to 0$ and comparing exponents, we see that no inequality of the form $\|S_z^\gamma f\|_{L^q} \leq C\|f\|_{L^p}$ is possible when $1/q < 2/p - 2 - \operatorname{Re} z$. Note that for a fixed z, $1/q = 2/p - 2 - \operatorname{Re} z$ is the equation of the line that intersects the segments BD and CD and is parallel to the line CE at height $(0, 0, \operatorname{Re} z)$. By duality we get that boundedness cannot hold when $2/q < 1/p - 1 - \operatorname{Re} z$. Again for a fixed z, $2/q = 1/p - 1 - \operatorname{Re} z$ is the equation of the line that intersects the segments BD and AD and is parallel to the line AE at height $(0, 0, \operatorname{Re} z)$. We have now proved that for a fixed z, $L^p \to L^q$ boundedness cannot hold when the point $(1/p, 1/q, \operatorname{Re} z)$ lies outside the triangle with vertices A′E′C′ where A′, E′, and C′ are the intersections of the lines BA, BE, and BC with the horizontal plane through $(0, 0, \operatorname{Re} z)$. This intersection is interesting to us only when $-3/2 \leq \operatorname{Re} z \leq -1$. The same argument applies to the degenerate case when the triangle A′E′C′ becomes the point B.

5. The Heisenberg group problem

In this section we discuss a similar issue on the Heisenberg group $\mathbb{H}^n$. $\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$ and with multiplication law $(z, t)(z', t') = (z + z', t + t' + 2\operatorname{Im} z \cdot \overline{z}')$ where $z \cdot \overline{z}' = \sum_{j=1}^n z_j\overline{z}'_j$. The norm of an element $u = (z, t) \in \mathbb{H}^n$ is defined by $|u| = (|z|^4 + |t|^2)^{1/4}$ and is homogeneous of degree 1 under the one-parameter group of dilations $r(z, t) \to (rz, r^2t)$. Let δ be the Dirac distribution in the t variable. Ricci and Stein [RS] considered the family of operators

$$S^\gamma f = f * \left[\Gamma\left(\frac{\gamma+1}{2}\right)^{-1} |z|^{\gamma-2n}\delta_{t=0}\right]$$

for $0 < \gamma \le 2n$, where $*$ is the Heisenberg group convolution. Define Γ to be the closed triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 1)$, and $(1/p_0, 1/q_0)$ minus the diagonal $\{(p, q): 1/p = 1/q\}$, where

$$p_0 = 1 + (2n+1)^{-1}, \qquad q_0 = 2n+2.$$

When $n = 1$, Ricci and Stein obtained $L^p \to L^q$ boundedness of S^γ for $(1/p, 1/q)$ in the interior of Γ and on a portion of its boundary, namely, when $6/5 \le p \le 2$. Christ [C2] proved $L^p \to L^q$ boundedness for all boundary points of Γ that do not lie on the diagonal for all $n \ge 1$. Furthermore, an example given in [C2] shows that no boundedness result can hold outside the closure of Γ. (The singular integral case $\gamma = 0$ has been treated by Geller and Stein [GSt].)

In this section we prove a similar result as in [C2] for an analytic family of fractional integrals S_w^γ, in which the operators S^γ can be embedded. The kernels of S_w^γ are obtained by taking $-w-1$ derivatives transverse to $\mathbb{C}^n$, and doing so in a dilation invariant way. Again the analytic family is defined in such a way as to satisfy $S_{-1}^\gamma = S^\gamma$. Fix a real smooth even nonnegative compactly supported bump function ψ equal to 1 in a neighborhood of 0. Our analytic family S_w^γ is given by convolution with the distribution

$$K_w^\gamma(z, t) = |z|^{\gamma-2n-2w-2}\Gamma\left(\frac{w+1}{2}\right)^{-1}|t|^w\psi(t/|z|^2).$$

For $\operatorname{Re} w > -1$, one can define S_w^γ as

$$\iint f(z-z', t-u|z'|^2 - 2\operatorname{Im} z\cdot z')|u|^w\psi(u)\,du|z'|^{\gamma-2n}\,dz'.$$

By analytic continuation, S_w^γ can be defined to be a distribution-valued entire function of w with the property $S_{-1}^\gamma = S^\gamma$. Our second result describes the exact range of $(1/p, 1/q, z, \gamma)$ for which S_z^γ maps $L^p(\mathbb{H}^n)$ to $L^q(\mathbb{H}^n)$ when $\operatorname{Re}\gamma > 0$. Since by homogeneity considerations, such a boundedness result can only hold when $1/p - 1/q = \operatorname{Re}\gamma/2(n+1)$, it is enough to describe the possible range of p, q, and w for which S_w^γ maps L^p to L^q. The precise statement of the theorem is

Theorem 2. *For* $\operatorname{Re}\gamma > 0$, *the analytic family of fractional integrals* S_w^γ *maps* L^p *to* L^q *if and only if* $(1/p, 1/q, \operatorname{Re} w)$ *lies* on or vertically above *the interiors of the faces* BCD *and* ABD *union the segment* BD$-\{$B$\}$ *of the tetrahedron* ABCD *with vertices* $\mathrm{A} = (0, 0, -1)$, $\mathrm{B} = (1/2, 1/2, -n-1)$, $\mathrm{C} = (1, 1, -1)$, *and* $\mathrm{D} = (1, 0, 0)$. (*See Figure* 2.)

Proof. Again, we will use Theorem 2 in [C2] to treat part of the kernel of S_w^γ. The proof of this theorem is similar to the proof of Theorem 1. The L^2 boundedness follows from the work of Geller and Stein [GSt]. They prove that if $\Phi(z, t) \in C^\infty(\mathbb{H}^n - \{0\})$, homogeneous of degree 0, $0 \le \Phi \le 1$ and such that for some $C_0 > 0$, $\Phi(z, t) = 1$ if $|t| \le C_0|z|^2$, $\Phi(z, t) = 0$ if $|t| \ge C_0|z|^2$, then $\mathbb{H}^n$-convolution with the distribution $\Gamma(\gamma'/2)^{-1}\Phi(z, t)|z|^{-2(n+\gamma')}|t|^{-1+\gamma'}$ maps L^2 to L^2 with bounds of admissible growth if and only if $\operatorname{Re}\gamma' \ge -n$. (In their paper γ' is denoted by γ.) Setting $\gamma' = w+1$ and $\Phi(z, t) = \psi(t/|z|^2)$, we get that when $\operatorname{Re}\gamma = 0$, S_w^γ maps L^1 to L^2 if and only if $\operatorname{Re} w \ge -(n+1)$.

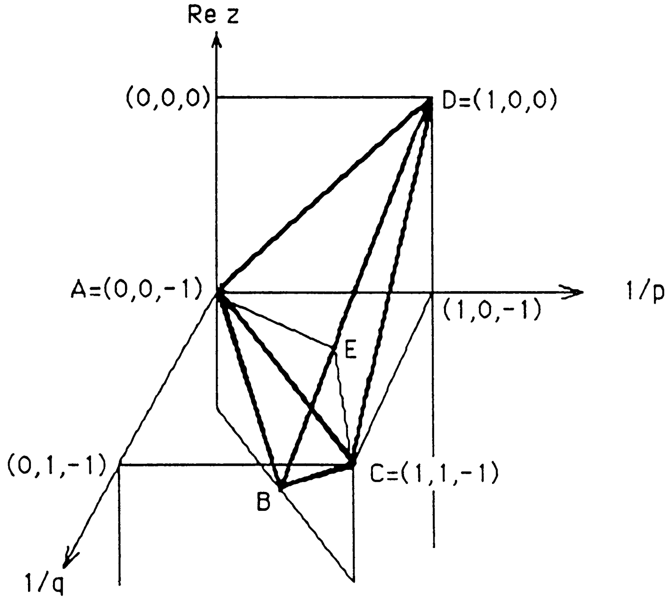

FIGURE 2. $B = (2n + 1/2(n+1), 1/2(n+1), -1)$, $E = (1/2, 1/2, -n-1)$

Also, one can easily see that when $\operatorname{Re}\gamma = 2(n+1)$, S_w^γ maps L^1 to L^∞ if and only if $\operatorname{Re} w \geq 0$. Analytic interpolation gives that for $(1/p, 1/q, \operatorname{Re} w) \in \mathrm{BD}$, S_w^γ maps L^p to L^q. (Here $q = p'$.)

Let $E = (1/p_0, 1/q_0, -1)$. Our proof will be complete by interpolation if we can show that for $(1/p, 1/q, -1)$ in the segment $\mathrm{AE} - \{\mathrm{A}\}$ and $\operatorname{Re} w = -1$, S_w^γ maps L^p to L^q with bounds of admissible growth. To prove this, let us fix p, q, γ, and w with $\operatorname{Re} w = -1$, set $K_{w,1}^\gamma = K_w^\gamma \chi_{|z|\leq 1}$, and define a distribution $H_w = \Gamma(\frac{w+1}{2})^{-1}|t|^w \psi(t)\delta_{z=0}$. By $k_{w,\gamma}$ we will denote the kernel $\delta_{t=0}|z|^{\gamma-2n-2w-2}\chi_{|z|\leq 1}$. The basic property of $k_{w,\gamma}$ is that it convolves L^p to L^q. This is because of the following inequalities:

$$\|f * k_{w,\gamma}\|_{L^q} = \left\| \int_{|z'|\leq 1} |f(z - z', t - 2\operatorname{Im} z \cdot \overline{z}')|\, |z'|^{\gamma-2n-2w-2} dz' \right\|_{L^q},$$

$$\left\| \int_{\mathbf{C}^n} |f(z - z', t - 2\operatorname{Im} z \cdot \overline{z}')|\, |z'|^{\gamma-2n} dz' \right\|_{L^q} \leq C_{p,\gamma}\|f\|_{L^p}.$$

The last inequality follows from Theorem 2 in [C2]. We need the following

Lemma. $K_{w,1}^\gamma = k_{w,\gamma} * H_w + \zeta(z, t)$ *where* $\zeta(z, t)$ *satisfies*

$$|\zeta(z, t)| \leq C_w \chi_{|t|\geq |z|^2} |z|^{\operatorname{Re}\gamma-2n-2} (|t|/|z|^2)^{-1}.$$

Proof. The proof of the lemma follows from an easy calculation. We have that

$$(f * k_{w,\gamma} * H_w)(z, t)$$
$$= \int_{|z'|\leq 1} \int f(z - z', t - t' - 2\operatorname{Im} z \cdot \overline{z}')|z'|^{\gamma-2n-2w-2} \frac{|t'|^w}{\Gamma\left(\frac{w+1}{2}\right)} \psi(t') dt'\, dz'.$$

It follows that the difference $f * K_{w,1}^{\gamma} - f * k_{w,\gamma} * H_w$ is equal to $f * \zeta$, where

$$\zeta(z, t) = \Gamma\left(\frac{w+1}{2}\right)^{-1} |z|^{\gamma-2n-2w-2}|t|^{w}\left[\psi\left(\frac{t}{|z|^2}\right) - \psi(t)\right]\chi_{|z|\le 1}.$$

Since ψ vanishes near 0, the function ζ above is supported in $|t| \ge |z|^2$ and the required estimate for ζ follows.

To prove the theorem, it is enough to consider $K_{w,M}^{\gamma} = K_w^{\gamma}\chi_{|z|\le M}$ for all $M > 0$ and prove that they convolve L^p to L^q with bounds uniform in M. By homogeneity, it suffices to prove that $K_{w,1}^{\gamma}$ convolves L^p to L^q. This will be a consequence of the lemma. First note that for all $1 < q < \infty$, $\|f * H_w\|_{L^q} \le C_{w,q}\|f\|_{L^q}$ is a consequence of the Hörmander multiplier theorem. It then follows that

$$\|(f * k_{w,\gamma}) * H_w\|_{L^q} \le C_{q,w}\|f * k_{w,\gamma}\|_{L^q} \le C_{p,w,\gamma}\|f\|_{L^p}.$$

It remains to control $\|f * \zeta\|_{L^q}$ by $C_{p,w,\gamma}\|f\|_{L^p}$. Similar argument as in §3 gives that $\zeta \in L^{r,\infty}$ where $r = (2n+2)/(2n+2-\operatorname{Re}\gamma)$. Young's inequality gives that convolution with ζ maps L^p to L^q, where p, q, and r are related as in $1/r + 1/p = 1 + 1/q$, which is equivalent to $1/p - 1/q = \operatorname{Re}\gamma/2(n+1)$. This concludes the proof of our claim. The proof of Theorem 2 follows from interpolation. The fact that Theorem 2 describes the exact range of p, q, w, and γ for which S_w^{γ} maps L^p to L^q when $\gamma > 0$, follows from an argument similar to the one given in §4. We omit the details.

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References

[BL] J. Bergh and J. Löfström, *Interpolation spaces, an introduction*, Springer Verlag, Berlin, Heidelberg, and New York, 1976.

[C1] M. Christ, *Weak type* (1, 1) *bounds for rough operators*, Ann. of Math. (2) **128** (1988), 19–42.

[C2] ——, *Endpoint bounds for singular fractional integral operators*, preprint.

[GS] I. M. Gelfand and G. E. Shilov, *Generalized functions*, Vol. I, Academic Press, New York and London, 1964.

[GSt] D. Geller and E. M. Stein, *Estimates for singular convolution operators on the Heisenberg group*, Math. Ann. **267** (1984), 1–15.

[G1] L. Grafakos, *Endpoint bounds for an analytic family of Hilbert transforms*, Duke Math. J. **62** (1991), 23–59.

[G2] ——, *Weak type estimates for a singular convolution operator on the Heisenberg group*, Trans. Amer. Math. Soc. **325** (1991), 435–452 .

[NSW] A. Nagel, E. M. Stein, and S. Wainger, *Differentiation in lacunary directions*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 1060–1062.

[O] D. Oberlin, *Convolution estimates for some measures on curves*, Proc. Amer. Math. Soc. **99** (1987), 56–60.

[RS] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals* III: *Fractional integration along manifolds*, MSRI preprint, 1988.

[S1] E. M. Stein, *Problems in harmonic analysis related to curvature and oscillatory integrals*, Proceedings of the International Congress of Mathematicians, Berkeley, CA, 1986.

[S2] ——, *Singular integrals and differentiability properties*, Princeton Univ. Press, Princeton, NJ, 1970.

[SWA] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

[Z] A. Zygmund, *Trigonometric series*, Vol. I, 2nd ed., Cambridge Univ. Press, London, 1959.

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