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6-TORSION AND HYPERBOLIC VOLUME

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ABSTRACT. The Kleinian group $\mathrm{PGL}(2, \mathcal{O}_3)$ is shown to have minimal covolume ($\approx 0.0846\ldots$) among all Kleinian groups containing torsion of order 6 (the associated hyperbolic orbifold is also the minimal volume cusped orbifold). This follows from: Any cocompact Kleinian group with torsion of order 6 has covolume at least $\frac{1}{9}$. As a consequence, any compact hyperbolic manifold with a symmetry of order 6 (with fixed points) has volume at least $\frac{4}{3}$. These results follow from new collaring theorems for torsion in a Kleinian group arising from our generalizations of the Shimizu-Leutbecher inequality.

1. INTRODUCTION

This paper is an application of the methods and results of our papers [4–7] concerning universal constraints on the geometry of Kleinian groups. For the definition of and basic results concerning Kleinian groups see the books of Beardon [2], Maskit [10], and Thurston [15]. Every Kleinian group acts as a group of isometries of hyperbolic 3-space $\mathbf{H}^3$. The orbit space, $\mathbf{H}^3/\Gamma$, of a Kleinian group is a *hyperbolic orbifold* (*hyperbolic manifold* if Γ is torsion free).

A fundamental problem is to determine the 3-dimensional hyperbolic orbifold of minimal volume. In the classical 2-dimensional case of Fuchsian groups, it is well known that the $(2, 3, 7)$ triangle group uniquely achieves this minimum. This fact has a number of important geometric consequences for the theory of Riemann surfaces. Lower bounds for the volume of all hyperbolic manifolds and orbifolds can be found in Myerhoff's work [12–14] and also our own [4, 7]. These bounds are far from sharp. Some sharp bounds are known in specific instances. Namely the minimal cusped hyperbolic orbifold [13] and cusped hyperbolic manifold [1] are both known (*cusped* here means that the orbit space is noncompact and has finite volume). See, too, [9] for the minimal hyperbolic manifold with totally geodesic boundary.

We expect that the Kleinian group of minimal covolume and torsion of order 2, 3 and one of 4 or 5 actually realises the minimum volume of all hyperbolic orbifolds. This is because in the minimal hyperbolic orbifold we believe the

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maximal point stabilizers to be the spherical triangle groups (see especially [7] where we establish inequalities that bound the order of the torsion in the minimal hyperbolic orbifold). A natural question is to identify the Kleinian group of minimal covolume and torsion of other orders. Our main result is identification of this group for torsion of order 6. Our methods are close to those of Meyerhoff [12] and rely on new collaring theorems derived from our generalization of the Shimizu-Leutbecher inequality [6] for torsion of order 6.

First a few definitions. The *singular set* of a hyperbolic orbifold is the projection into the orbit space of the fixed points of the elliptic (torsion) elements of Γ (it is the branch locus of the projection). We say that an orbifold $Q = \mathbf{H}^3/\Gamma$ has *singular set of degree* p if there is a maximal cyclic subgroup of order p in the Kleinian group Γ (the orbifold fundamental group).

1.1. **Theorem.** *The hyperbolic orbifold of minimal volume and singular set of degree* 6 *is the cusped orbifold* $Q_6 = \mathbf{H}^3/\mathrm{PGL}(2, O_3)$, *where* O_3 *is the ring of integers in* $Q(\sqrt{-3})$.

The orbifold Q_6 is also the orientable double cover of the tetrahedral orbifold with Coxeter diagram $\circ - \circ - \circ \equiv \circ$. The volume is ($\frac{1}{24}$ the volume of the ideal regular tetrahedron) ≈ 0.0846. This orbifold is also the minimal volume cusped hyperbolic orbifold. Theorem 1.1 follows from

1.2. **Theorem.** *Any compact hyperbolic orbifold with singular set of degree* 6 *has volume at least* $\frac{1}{9}$.

A hyperbolic manifold M has a *symmetry of order* p if there is an isometry of M with fixed points and whose order is p. As a consequence of the above we have

1.3. **Corollary.** *Any compact hyperbolic manifold with a symmetry of order* 6 *has volume at least* $\frac{4}{3}$.

Note that the bound of Corollary 1.3 is larger than the volumes of certain known hyperbolic manifolds. We obtain results related to Theorem 1.2 and Corollary 1.3 for more general torsion in [7]. Our method there, as it is here, is to estimate the volume of a tubular neighbourhood about a component of the singular set.

2. Precisely invariant collars

We begin by introducing some notation. For a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc = 1,$$

we set $\mathrm{tr}(f) = a + d$ and define

$$\beta(f) = \mathrm{tr}^2(f) - 4;$$

for Möbius transformations f and g we let

$$\gamma(f, g) = \mathrm{tr}[f, g] - 2.$$

The numbers $\beta(f)$, $\beta(g)$, and $\gamma(f, g)$ are well defined and determine the two generator group $\langle f, g\rangle$ uniquely up to conjugacy when $\gamma(f, g) \neq 0$ [3].

We say an elliptic element f of order p is *primitive* if $\beta(f) = -4\sin^2(\pi/p)$; that is, if the multiplier of f is $e^{\pm 2\pi i/p}$.

The main tool used in the computation of the minimal cusped manifolds and orbifolds is the Shimizu-Leutbecher inequality guaranteeing the existence of precisely invariant horospheres. One of our main tools is the following generalization of that result for torsion of order 6. See [6, Theorem 3.4].

2.1. **Lemma.** *Let $\langle f, g\rangle$ be discrete and nonelementary. If f is elliptic of order 6, then*

$$|\gamma(f, g)| \geq 1 \tag{2.2}$$

and

$$|\gamma(f, g) + 1| \geq 1 \quad \textit{or} \quad \gamma(f, g) = -1. \tag{2.3}$$

To give the reader a flavour of our argument we sketch a proof for the following result. See [6].

2.4. **Lemma.** *Let $\langle f, g\rangle$ be discrete and nonelementary with no parabolic elements. If f and g are both elliptic of order 6, then*

$$|\gamma(f, g)| \geq \sqrt{2}. \tag{2.5}$$

Proof. Let

$$D_1 = \{z \in \mathbb{C} : |z| < 1\}, \qquad D_2 = \{z \in \mathbb{C} : |z + 1| < 1\}, \qquad \psi(z) = z(1+z).$$

Then $\gamma(f, g) \notin \psi(D_1)$. For otherwise by [3, Lemma 2.2] we could choose γ with $|\gamma| < 1$ and an elliptic h of order 2 such that $\gamma = \gamma(f, h)$ and $\gamma(f, g) = \gamma(\gamma+1)$; then $\langle f, g\rangle$ would be conjugate to $\langle f, hfh^{-1}\rangle$, an index two subgroup of $\langle f, h\rangle$, and $\langle f, h\rangle$ would be discrete and nonelementary contradicting (2.2). Next $\gamma(f, g) \neq -1$ since otherwise f and gfg^{-1} would have a common fixed point and $[f, gfg^{-1}]$ would be parabolic. Thus $\gamma(f, g) \notin \psi(D_1) \cup D_2$ by (2.3) and (2.5) follows from elementary geometric considerations.

We point out that the possibility $[f, gfg^{-1}]$ is parabolic can arise in general, see [6, 11]. It is here that the hypothesis concerning the absence of parabolic elements (we will later assume the orbifold is closed) enables us to significantly improve our lower bound on $|\gamma(f, g)|$. We next recall another result of [6, Lemma 4.2].

2.6. **Lemma.** *Suppose that f and g have disjoint pairs of fixed points, and let $\beta = \beta(f)$, $\beta' = \beta(g)$, and $\gamma = \gamma(f, g)$. Then*

$$\sinh^2(\delta \pm i\theta) = 4\gamma/\beta\beta' \tag{2.7}$$

where δ is the hyperbolic distance between the axes of f and g, and θ is the dihedral angle between the two spheres containing the common perpendicular and one of these axes, respectively.

The following lemma is a simple modification of a result of Zagier as used by Meyerhoff [12]. We include a proof because it is important in what follows.

2.8. **Lemma.** *For each $x \in (0, \pi/2\sqrt{3}]$ and $\theta \in [0, 2\pi]$*

$$\min_{j\geq 1}\min_{k}\left\{\cosh(jx) - \cos\left(j\theta + \frac{k\pi}{3}\right)\right\} \leq \cosh\left(\sqrt{\frac{2\pi x}{3\sqrt{3}}}\right) - 1.$$

Proof. Let $u = 3\theta/\pi$ and $v = 3x/\pi$. Then, by a standard argument using the modular group [12, p. 1045], there are integers m and n such that

$$(nu - m)^2 + n^2v^2 \leq \frac{2v}{\sqrt{3}}, \qquad n \geq 1. \tag{2.9}$$

Thus,

$$\begin{aligned}\min_{j\geq 1}\min_{k}\left\{\cosh(jx) - \cos\left(j\theta + \frac{2k\pi}{6}\right)\right\} &\leq \cosh(nx) - \cos\left(n\theta - \frac{2m\pi}{6}\right)\\ &= \sum_{r=1}^{\infty}\frac{1}{(2r)!}\left[(nx)^{2r} - (-1)^r\left(n\theta - \frac{m\pi}{3}\right)^{2r}\right]\\ &\leq \sum_{r=1}^{\infty}\frac{1}{(2r)!}\left[(nx)^2 + \left(n\theta - \frac{m\pi}{3}\right)^2\right]^r,\\ &\leq \sum_{r=1}^{\infty}\frac{1}{(2r)!}\left(\sqrt{\frac{2\pi x}{3\sqrt{3}}}\right)^{2r} = \cosh\left(\sqrt{\frac{2\pi x}{3\sqrt{3}}}\right) - 1,\end{aligned}$$

by (2.9).

Let Γ be a Kleinian group. We say $f \in \Gamma$ is *simple* if the axis of f is precisely invariant. That is if for all $g \in \Gamma$, either

$$g(\operatorname{axis}(f)) = \operatorname{axis}(f) \quad \text{or} \quad g(\operatorname{axis}(f)) \cap \operatorname{axis}(f) = \varnothing. \tag{2.10}$$

Our results are based on the following three collaring lemmas.

2.11. **Lemma.** *Let Γ be a Kleinian group without parabolics and suppose $f \in \Gamma$ is elliptic of order* 6. *Then there is a precisely invariant collar about* $\operatorname{axis}(f)$ *of radius r where*

$$\sinh^2(r) \geq 2^{1/4} - \tfrac{1}{2} \approx 0.6892\ldots.$$

Proof. We may assume f is primitive. The classification of the elementary Kleinian groups [10] implies f is simple. Let $g \in \Gamma$. If $g(\operatorname{axis}(f)) = \operatorname{axis}(f)$, then any collar about $\operatorname{axis}(f)$ is precisely invariant. Otherwise set $h = gfg^{-1}$. Then $\langle f, h\rangle$ is discrete and nonelementary. Hence by (2.5) $|\gamma(f, h)| \geq \sqrt{2}$. From (2.7) we have

$$|\sinh^2(\delta + i\theta)| = \left|\frac{4\gamma}{\beta\beta'}\right| = 4|\gamma| \geq 4\sqrt{2}$$

whence

$$\sinh^2(\delta) \geq 4\sqrt{2} - 1.$$

The distance between $\operatorname{axis}(f)$ and $\operatorname{axis}(h)$ is δ and as $\operatorname{axis}(h) = g(\operatorname{axis}(f))$ any collar of radius $r = \delta/2$ is mapped off itself by g. The result follows from the hyperbolic trigonometric identity,

$$\sinh^2(u) = \tfrac{1}{2}(\sqrt{\sinh^2(2u) + 1} - 1).$$

2.12. **Lemma.** *Let Γ be a Kleinian group. Suppose that $f \in \Gamma$ is elliptic of order* 6, *that $g \in \Gamma$ is loxodromic with translation length x and that* $\operatorname{axis}(g) =$

axis(f). *Then there is a precisely invariant collar about* axis(f) *of radius* r *where*

$$\sinh^2(r) \geq \frac{1}{2}\left(\frac{\sqrt{3-2\alpha(x)}}{\alpha(x)-1} - 1\right) \geq 0 \tag{2.13}$$

for $x \in (0, \alpha^{-1}(\sqrt{2})]$ *and*

$$\alpha(x) = \cosh\left(\sqrt{\frac{2\pi x}{3\sqrt{3}}}\right). \tag{2.14}$$

Proof. The hypotheses imply f is simple. So too is g, which we may assume is also primitive. Suppose that $\beta(g) = -4\sinh^2(x/2 + i\theta/2)$. Then for $j \geq 1$ and any k,

$$\beta(g^j f^k) = -4\sinh^2\left(\frac{jx}{2} + i\left(\frac{j\theta}{2} + \frac{k\pi}{6}\right)\right). \tag{2.15}$$

Hence

$$\begin{aligned} |\beta(g^j f^k)| &= 4\left(\sinh^2\left(\frac{jx}{2}\right) + \sin^2\left(\frac{j\theta}{2} + \frac{k\pi}{6}\right)\right) \\ &= 2\left(\cosh(jx) - \cos\left(j\theta + \frac{k\pi}{3}\right)\right). \end{aligned}$$

Now $\alpha^{-1}(\sqrt{2}) < \pi/2\sqrt{3}$. Therefore by replacing g by $g^j f^k$ for suitable j and k, we may assume by Lemma 2.8 that

$$|\beta(g)| \leq 2\cosh\left(\sqrt{\frac{2\pi x}{3\sqrt{3}}}\right) - 2 = 2(\alpha(x) - 1).$$

Then by (2.7), for any $h \in \Gamma$ with $h(\text{axis}(g)) \cap \text{axis}(g) = \varnothing$,

$$\sinh^2(\delta) \geq \frac{4|\gamma(g, hgh^{-1})|}{|\beta(g)\beta(hgh^{-1})|} - 1, \tag{2.16}$$

where δ is the hyperbolic distance between axis(g) and $h(\text{axis}(g))$. Note that $\beta(hgh^{-1}) = \beta(g)$. Now Jørgensen's inequality [8] implies

$$|\gamma(g, hgh^{-1})| \geq 1 - |\beta(g)| \tag{2.17}$$

since the group $\langle g, hgh^{-1}\rangle$ is discrete and nonelementary. Hence

$$\sinh^2(\delta) \geq \frac{4(1 - |\beta(g)|)}{|\beta(g)|^2} - 1 \geq \frac{3 - 2\alpha(x)}{(\alpha(x) - 1)^2} - 1. \tag{2.18}$$

The result follows again by the above hyperbolic trigonometric identity.

2.19. **Lemma.** *Let* Γ *be a Kleinian group. Suppose that* $f \in \Gamma$ *is elliptic of order* 6, *that* $g \in \Gamma$ *is loxodromic with translation length* x *and that* axis(g) = axis(f). *Then there is a precisely invariant collar about* axis(g) *of radius* r *where*

$$\sinh^2(r) \geq \frac{\sqrt{1 - \eta(x)}}{\eta(x)} - \frac{1}{2} \geq 0 \tag{2.20}$$

for $x \in (0, \eta^{-1}(2(\sqrt{2}-1))]$,

$$\sinh^2(r) \geq \frac{1}{\sqrt{\eta(x)}} - \frac{1}{2} \geq 0 \tag{2.21}$$

for $x \in (0, \eta^{-1}(4)]$ *and where*

$$\eta(x) = 4\left(\sinh^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{\pi}{12}\right)\right). \tag{2.22}$$

Proof. Choose $k \in \mathbb{Z}$ so that $|\theta/2 + k\pi/6| \leq \pi/12$, where θ is the rotation angle of g. Then by (2.15)

$$\begin{aligned} |\beta(gf^k)| &= 4\left(\sinh^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{\theta}{2} + \frac{k\pi}{6}\right)\right) \\ &\leq 4\left(\sinh^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{\pi}{12}\right)\right) = \eta(x). \end{aligned} \tag{2.23}$$

Thus by replacing g by gf^k we may assume that $|\beta(g)| \leq \eta(x)$. Suppose that $h \in \Gamma$ with $h(\operatorname{axis}(g)) \cap \operatorname{axis}(g) = \varnothing$. If δ is the hyperbolic distance between $\operatorname{axis}(g)$ and $h(\operatorname{axis}(g))$, then as in (2.16) and (2.17)

$$\sinh^2(\delta) \geq \frac{4|\gamma(g, hgh^{-1})|}{|\beta(g)|\,|\beta(hgh^{-1})|} - 1 \geq 4\frac{1-\eta(x)}{\eta^2(x)} - 1$$

and (2.20) follows. Next since $\langle f, hgh^{-1}\rangle$ is discrete and nonelementary,

$$\sinh^2(\delta) \geq \frac{4|\gamma(f, hgh^{-1})|}{|\beta(f)|\,|\beta(hgh^{-1})|} - 1 \geq \frac{4}{\eta(x)} - 1$$

by (2.2), and we obtain (2.21).

3. Proofs

Let Γ be a Kleinian group and suppose that Γ contains no parabolic elements. If $f \in \Gamma$ is elliptic of order 6, then f is simple by the classification of the elementary Kleinian groups. We may assume that f is primitive. Suppose there is a collar of radius 4 about $\operatorname{axis}(f)$. As the volume of this collar is infinite, there is a loxodromic $g \in \Gamma$ with $\operatorname{axis}(g) = \operatorname{axis}(f)$. We may assume g is primitive. Let x be the translation length of g. The volume of the solid tube of length x and radius r about $\operatorname{axis}(f)$ is

$$V(x) = \pi x \sinh^2(r). \tag{3.1}$$

Note that g moves this tube off itself. We want to find the minimum of our lower bounds for $V(x)$ over all values of x and associated collar radius r. We do this by breaking the situation up into three cases covered by our three collaring lemmas.

1. The diophantine analysis of Lemma 2.12 and (3.1) imply

$$V(x) \geq V_1(x) = \frac{\pi x}{2}\left(\frac{\sqrt{3-2\alpha(x)}}{\alpha(x)-1} - 1\right) \tag{3.2}$$

for $x \in (0, \alpha^{-1}(\sqrt{2})]$ where $\alpha(x)$ is defined by (2.14).

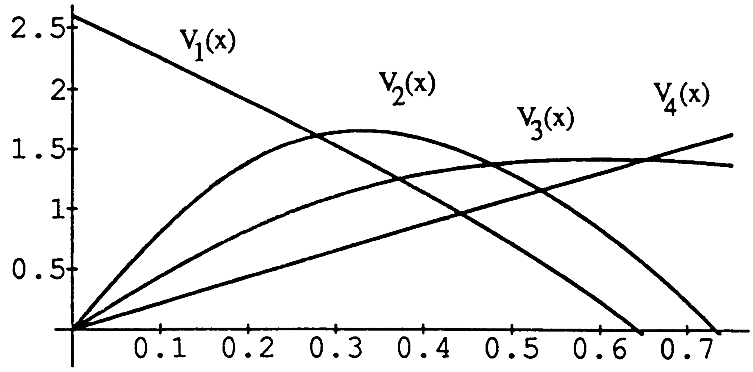

FIGURE 1. Lower bounds for the function $V(x)$.

2. Lemma 2.19 and (3.1) imply

$$V(x) \geq V_2(x) = \frac{\pi x}{2}\left(\frac{2\sqrt{1-\eta(x)}}{\eta(x)} - 1\right) \tag{3.3}$$

for $x \in (0, \eta^{-1}(2(\sqrt{2}-1))]$ and

$$V(x) \geq V_3(x) = \frac{\pi x}{2}\left(\frac{2}{\sqrt{\eta(x)}} - 1\right) \tag{3.4}$$

for $x \in (0, \eta^{-1}(4)]$ where $\eta(x)$ is defined by (2.22).

3. Lemma 2.11 and (3.1) imply

$$V(x) \geq V_4(x) = \pi x(2^{1/4} - \tfrac{1}{2}) \tag{3.5}$$

for $x \in (0, \infty)$.

The graph of the functions $V_i(x)$, $i = 1, 2, 3, 4$, appears in Figure 1.

We need the following lemma.

3.6. **Lemma.** *For each* $a \in (0, \infty)$, *the function*

$$V_1(x) = \frac{\pi}{2}x\left(\frac{\sqrt{3-2\alpha(x)}}{\alpha(x)-1} - 1\right)$$

is decreasing on the interval $(0, \alpha^{-1}(3/2))$ *where* $\alpha(x) = \cosh\sqrt{ax}$.

Proof. By definition

$$x\alpha'(x) = \frac{1}{2}\sqrt{ax}\sinh\sqrt{ax} = \frac{1}{2}\sum_{r=1}^{\infty}\frac{(ax)^r}{(2r-1)!} \geq \sum_{r=1}^{\infty}\frac{(ax)^r}{(2r)!} = \alpha(x) - 1,$$

and we obtain

$$\begin{aligned}\frac{2}{\pi}V_1'(x) &= \frac{x\alpha'(x)(\alpha(x)-2) + (3-2\alpha(x))(\alpha(x)-1)}{\sqrt{3-2\alpha(x)}(\alpha(x)-1)^2} \\ &\leq -\frac{1}{\sqrt{3-2\alpha(x)}} - 1 \leq -2\end{aligned}$$

for $x \in (0, \alpha^{-1}(3/2))$.

Given Lemma 3.6 it is a relatively straightforward matter to estimate lower bounds for the intersection points of the graphs of V_1 and V_2, V_2 and V_3, and then V_3 and V_4. An easy lower bound for these values is 1.37 establishing

3.7. **Proposition.** $V(x) \geq 1.37$.

As f and g are primitive, the stabilizer of the collar constructed in each case contains $\langle f, g\rangle$ as an index one or two subgroup (possibly there is an elliptic h of order two interchanging the fixed points of f and $\langle f, h\rangle$ is dihedral of order 12). Therefore the order of the stabilizer of the solid tube constructed above is at most 12. Whence, combining (3.2)–(3.5) we obtain

$$\mathrm{Vol}(\mathbf{H}^3/\Gamma) \geq V(x)/12 \geq 0.114.$$

This proves Theorem 1.2.

Theorem 1.1 now follows directly as the cusped hyperbolic orbifold mentioned has minimal volume among all such orbifolds [13] and has volume less than our lower bound for the compact case.

Next, if $M = \mathbf{H}^3/\Gamma$ is a hyperbolic manifold and f is an isometric symmetry of M of order 6 with fixed points, then every component of the singular set of the hyperbolic 3-orbifold $M/\langle f\rangle$ has degree 6. In particular, the universal orbifold covering Γ' only has torsion of order 6 (actually there is a conjugacy class of 6-torsion for every component of the singular set). Hence the stabilizer of the solid tube constructed in the proof of Theorem 1.2 is precisely invariant under (the torsionfree) subgroup Γ of Γ' and so projects isometrically into M. Its volume yields a lower bound for the volume of M. This proves Corollary 1.3.

3.8. *Remarks.* As in [4, 14] we could improve our lower bound on the minimal volume of compact hyperbolic orbifolds with singular set of degree 6 by using a packing argument (the orbit of a precisely invariant set gives a packing of hyperbolic 3-space by congruent objects). However, the situation is not quite so simple as in [14] since the packing is not with rounds balls, but with solid tubes.

Finally, in [7] we give examples of Kleinian groups Γ_p, $p \geq 7$, containing a primitive elliptic of order p and of cofinite volume. We believe these examples to be of minimal covolume among all Kleinian groups with this property. They are extremal groups for the distance between conjugate elliptic axes of order p and consequently exhibit the sharpness of the collaring theorems we establish in [6] even among the cofinite volume Kleinian groups. The accompanying estimates we make on the volume of all such hyperbolic orbifolds imply that if $p \geq 12$, then the volume of any hyperbolic orbifold with singular set of degree p exceeds the covolume of $\mathrm{PGL}(2, \mathcal{O}_3)$. This in turn implies that Theorem 1.1 is true as soon as 6 divides the degree of the singular set.

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