PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 3, March 1993

REMARKS ON THE TWO WELL PROBLEM WITH ROTATIONS

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(Communicated by Barbara L. Keyfitz)

ABSTRACT. A conjecture of I. Fonseca on the form of minimizers of a double-well problem arising in the study of elastic crystal microstructure is answered to be true.

In the study of continuum models for phase transitions in elastic solids, Fonseca [1] employs a penalization method to a double-well model with rotations, showing that a sequence of minimizers of the approximation problem admits a subsequence converging weakly to a solution of the unperturbed problem of the form

$$\nabla u(x) = R(x)(B + \chi_E(x)\mathbf{a}\otimes\mathbf{n}) \quad \text{in } \Omega \subset \mathbb{R}^3 \tag{1}$$

subject to

$$\int_\Omega \nabla u\, dx = \operatorname{meas}(\Omega)(B + \theta\mathbf{a}\otimes\mathbf{n}), \quad 0 < \theta < 1, \qquad \int_\Omega u\, dx = m, \tag{2}$$

where $R(x) \in \mathrm{SO}(3)$ is a rotation, B is a 3×3 matrix with $\det B > 0$, ∇u, R and $\chi_E \in BV(\Omega) \cap L^\infty(\Omega)$, $u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$, ∇u is the gradient of u, and χ_E is the characteristic function of some measurable set $E \subset \Omega$. The main conjecture in [1] is that $R = I$—the unit matrix. The conjecture is confirmed if the set E determines a partition of Ω into countably many open, strongly Lipschitz connected domains [1, Theorem 5.8], or $\mathbf{a}$ is parallel to $B^{-\mathrm{T}}\mathbf{n}$. In this note we prove that Fonseca's conjecture is true, that is,

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be open and bounded; suppose that $u \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ is such that*

$$\nabla u(x) = R(x)(B + \chi_E(x)\mathbf{a}\otimes\mathbf{n}) \quad in\ \Omega,$$

$$\int_\Omega \nabla u\, dx = \operatorname{meas}(\Omega)[B + \theta\mathbf{a}\otimes\mathbf{n}], \qquad 0 < \theta < 1,$$

where $R \in \mathrm{SO}(3)$ a.e.; $\det B \neq 0$, $\mathbf{a}, \mathbf{n} \in \mathbb{R}^3$, $\mathbf{a} \neq 0$, $|\mathbf{n}| = 1$; $E \subset \Omega$ is measurable. Then $R = I$ a.e.

Proof. Since $\nabla u(x) = R(x)(I + \chi_E(x)\mathbf{a}\otimes(B^{-\mathrm{T}}\mathbf{n}))B$, by a change of coordinates we may assume that

$$\nabla u = R'(x)(I + \chi_E\mathbf{a}'\otimes\mathbf{n}'), \qquad \int_\Omega \nabla u\, dx = \operatorname{meas}(\Omega)[I + \theta\mathbf{a}'\otimes\mathbf{n}]$$

Received by the editors April 22, 1991 and, in revised form, June 19, 1991.
1991 *Mathematics Subject Classification*. Primary 49B36, 49A50, 73C50.

where $R'(x) = R(B^{-1}x)$, and we may find some $Q \in \mathrm{SO}(3)$ such that

$$Q\mathbf{n}' = \mathbf{e}_1, \quad Q\mathbf{a}' = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2, \qquad \mathbf{e}_1 = (1, 0, 0)^{\mathrm{T}}, \quad \mathbf{e}_2 = (0, 1, 0)^{\mathrm{T}},$$
$$Q\nabla u Q^{\mathrm{T}} = QR(x)Q^{\mathrm{T}}[QIQ^{\mathrm{T}} + \chi_E Q\mathbf{a}' \otimes (Q^{\mathrm{T}}\mathbf{n}')].$$

Hence by a further change of coordinates, we may assume that

$$\begin{aligned} &\nabla u = R(x)[I + \chi_E \alpha e_1 \otimes e_1 + \beta e_2 \otimes e_1], \qquad \beta \neq 0, \\ &\int_\Omega \nabla u\, dx = \mathrm{meas}(\Omega)[I + Q(\alpha e_1 \otimes e_1 + \beta e_2 \otimes e_1)]. \end{aligned} \tag{3}$$

We have

$$\nabla u = (\nabla u_1, \nabla u_2, \nabla u_3)^{\mathrm{T}}, \quad R = (R_1, R_2, R_3)^{\mathrm{T}}, \quad R_i = (R_{i1}, R_{i2}, R_{i3})^{\mathrm{T}},$$

and so

$$\int_\Omega \nabla u_3\, dx = \int_\Omega R_3\, dx = \mathrm{meas}(\Omega)(0, 0, 1)^{\mathrm{T}}.$$

Since $|R_3| = 1$, a.e. we deduce that $R_{33} = 1$ a.e. $R_{31} = R_{32} = 0$ a.e.

Since $R \in \mathrm{SO}(3)$, we have

$$R(x) = \begin{pmatrix} R_{11}(x) & R_{12}(x) & 0 \\ R_{21}(x) & R_{22}(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \in \mathrm{SO}(2) \tag{4}$$

a.e.

From (3), we have $\partial u_2/\partial x_2 = R_{22}(x)$, with $|R_{22}(x)| \leq 1$ a.e., and

$$\int_\Omega \frac{\partial u_2}{\partial x_2}\, dx = \mathrm{meas}(\Omega)[I + \theta(\alpha e_1 \otimes e_1 + \beta e_2 \otimes e_1)]_{22} = \mathrm{meas}(\Omega),$$

which implies $R_{22} = 1$ a.e. Therefore (4) implies $R_{11} = 1$, $R_{12} = R_{21} = 0$. □

Remark 1. Since $R = I$, we have $\nabla u = B + \chi_E \mathbf{a} \otimes \mathbf{n}$, and so

$$\mathrm{curl}\, \nabla u = 0 \quad \text{in the sense of distributions},$$

which holds if and only if the outward normal v to $\partial E \cap \Omega$ is parallel to $\mathbf{n}$.

Remark 2. The proof of the conjecture depends heavily on the constraints (2), which seems to be a very strong condition to ensure that $R(x) =$ Identity. I do not know what happens without this constraint. I conjecture that $R(x) = R_0 \in \mathrm{SO}(3)$ is a constant matrix in $\mathrm{SO}(3)$ without assuming (2). (Also see [1, Corollary 5.20].)

Acknowledgment

I am thankful to the referee for valuable suggestions.

References

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