A CHARACTERIZATION OF HYPERBOLIC MANIFOLDS

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Abstract. In this note we prove that a complex manifold $X$ is Kobayashi hyperbolic if and only if the space $\text{Hol}(\Delta, X)$ of holomorphic maps of the unit disk $\Delta$ into $X$ is relatively compact (with respect to the compact-open topology) in the space $C(\Delta, X^*)$ of continuous maps from $\Delta$ into the one-point compactification $X^*$ of $X$.

0. Introduction

Let $X$, $Y$ be (connected) complex manifolds; we shall denote by $\text{Hol}(X, Y)$ and $C(X, Y)$, respectively, the space of holomorphic and continuous maps from $X$ to $Y$, endowed with the compact-open topology.

Let $\delta_X : X \times X \to \mathbb{R}$ be defined by

$$\delta_X(z, w) = \inf \{ \omega(0, \zeta) \mid \exists \varphi \in \text{Hol}(\Delta, X) : \varphi(0) = z, \varphi(\zeta) = w \},$$

where $z, w \in X$, $\Delta$ is the unit disk in $\mathbb{C}$, and $\omega$ is the Poincaré distance on $\Delta$. Then the Kobayashi (pseudo)distance $k_X$ on $X$ is the largest (pseudo)distance dominated by $\delta_X$. More precisely, let us define an analytic chain $\alpha$ connecting two points $z_0, w_0 \in X$ as a sequence of pairs

$$\alpha = \{(\zeta_0, \varphi_0), \ldots, (\zeta_m, \varphi_m)\},$$

where $\zeta_0, \ldots, \zeta_m \in \Delta$, $\varphi_0, \ldots, \varphi_m \in \text{Hol}(\Delta, X)$ are such that $\varphi_0(0) = z_0, \varphi_j(\zeta_j) = \varphi_{j+1}(0)$ for $j = 0, \ldots, m - 1$, and $\varphi_m(\zeta_m) = w_0$. The length $\omega(\alpha)$ of the chain $\alpha$ is given by

$$\omega(\alpha) = \sum_{j=0}^{m} \omega(0, \zeta_j).$$

Then the Kobayashi (pseudo)distance $k_X$ between $z$ and $w \in X$ is given by

$$k_X(z, w) = \inf \{ \omega(\alpha) \},$$

where the infimum is taken with respect to all the analytic chains connecting $z$ to $w$. Since $X$ is connected, $k_X(z, w)$ is always finite, and it is clearly the largest pseudodistance dominated by $\delta_X$.

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The complex manifold $X$ is said to be hyperbolic if $k_X$ is an actual distance (i.e., $k_X(z, w) = 0$ implies $z = w$). In this case, the Kobayashi distance induces the original manifold topology on $X$ [B2]. There are many examples of hyperbolic manifolds; for instance, bounded domains in $\mathbb{C}^n$, hermitian manifolds with holomorphic sectional curvature bounded above by a negative constant, anything covered by (or covering) a hyperbolic manifold, and so on. For more on definitions and properties of the Kobayashi distance and hyperbolic manifolds, see [Ko2, Ko3, L, A].

The aim of this note is to prove a new characterization of hyperbolic manifolds. A few months before Kobayashi defined hyperbolic manifolds in [Ko1], Wu [W] (see also [B1]) introduced the notion of taut manifold. A complex manifold $X$ is called taut if $\text{Hol}(\Delta, X)$ is a normal family, that is, if every sequence $\{f_\nu\} \subset \text{Hol}(\Delta, X)$ has a subsequence $\{f_{\nu_k}\}$ that is either converging (uniformly on compact subsets) or compactly divergent (i.e., for every pair of compact sets $H \subset \Delta$ and $K \subset X$ one has $f_{\nu_k}(H) \cap K = \emptyset$ eventually).

It was immediately clear that a strong relation should exist between taut and hyperbolic manifolds; indeed, it turned out that every taut manifold is hyperbolic [Ki] and that every complete hyperbolic manifold (i.e., every manifold $X$ with $k_X$ complete as topological distance) is taut [Ki] (actually, it was conjectured that taut were equivalent to complete hyperbolic, until Rosay’s counterexample [R]).

We shall prove a characterization of hyperbolic manifolds showing explicitly this relationship. Let $X$ be a (connected) complex manifold, and let $X^*$ denote its one-point (or Alexandroff) compactification. It is easy to check that $X$ is taut iff $\text{Hol}(\Delta, X) \cup \{\infty\}$ is (closed and) compact in $C(\Delta, X^*)$, where $\infty$ denotes both the point at infinity of $X^*$ and the map of constant value $\infty$. Then we shall prove that $X$ is hyperbolic iff $\text{Hol}(\Delta, X)$ is relatively compact in $C(\Delta, X^*)$ (Theorem 1.3).

It would be nice to have a similar characterization for complete hyperbolic manifolds, that is, a characterization only in terms of topological (or uniform?) properties of $\text{Hol}(\Delta, X) \hookrightarrow C(\Delta, X^*)$; but as far as I know, this is an open question.

1. Hyperbolic manifolds

Let $(X, \tau)$ be a noncompact connected Hausdorff locally compact topological space; its one-point (or Alexandroff) compactification $(X^*, \tau^*)$ is the set $X \cup \{\infty\}$, where $\infty$ is a point not in $X$, endowed with the topology

$$\tau^* = \tau \cup \{(X \setminus K) \cup \{\infty\} | K \subset X \text{ compact}\}.$$

It is easy to check (see [K, 5.21]) that $(X^*, \tau^*)$ is a connected Hausdorff compact topological space, with $X$ as dense subspace. Furthermore, if $X$ is second countable, then so is $X^*$, which, therefore, is metrizable [K, 4.16]. In particular, if $Y$ is another locally compact metrizable second countable space (a manifold, for instance) then $C(Y, X^*)$ endowed with the compact-open topology is still metrizable and a subset of $C(Y, X^*)$ is compact iff it is sequentially compact.

A moment’s thought shows that a sequence $\{f_\nu\} \subset C(Y, X)$ is compactly divergent iff it converges, in $C(Y, X^*)$, to the constant map $\infty$; hence it is clear that a complex manifold $X$ is taut iff $\text{Hol}(\Delta, X)$ is relatively compact in $C(\Delta, X^*)$ and its closure is $\text{Hol}(\Delta, X) \cup \{\infty\}$, as mentioned in the introduction.
The proof of the characterization of hyperbolic manifolds we are after depends on the topological Ascoli-Arzelà theorem due to Kelley and Palais [K]. We recall the definitions involved.

Let \( X, Y \) be two topological spaces. A family \( \mathcal{F} \subset C(X, Y) \) is evenly continuous if for every \( x \in X, y \in Y \), and every neighbourhood \( U \) of \( y \) in \( Y \) there is a neighbourhood \( V \) of \( x \) in \( X \) and a neighbourhood \( W \) of \( y \) in \( Y \) such that for every \( f \in \mathcal{F} \)

\[
f(x) \in W \Rightarrow f(V) \subset U.
\]

In other words, points that are close to each other remain close in a uniform way under the action of elements of \( \mathcal{F} \); it is a topological version of equicontinuity.

Then the topological Ascoli-Arzelà theorem is

**Theorem 1.1** [K, 7.21]. Let \( X \) be a regular locally compact topological space and \( Y \) a regular Hausdorff topological space. Then a family \( \mathcal{F} \subset C(X, Y) \) is relatively compact in \( C(X, Y) \) iff it is evenly continuous and \( \{f(x) | f \in \mathcal{F}\} \) is relatively compact in \( Y \) for all \( x \in X \).

Since holomorphic maps contract the Kobayashi distance, \( \text{Hol}(\Delta, X) \) is clearly an equicontinuous family in every hyperbolic manifold \( X \). To get even continuity in \( C(\Delta, X^*) \) we need the following

**Lemma 1.2.** Let \( (X, d) \) be a noncompact, connected, locally compact metric space and \( (Y, d') \) another metric space. Then an equicontinuous family \( \mathcal{F} \subset C(Y, X) \) is evenly continuous as a subset of \( C(Y, X^*) \), and thus as a subset of \( C(Y, X) \), too.

**Proof.** Fix \( x_0 \in X^* \) and \( y_0 \in Y \); we must show that for any neighbourhood \( U \) of \( x_0 \) there are neighbourhoods \( V \) of \( y_0 \) and \( W \) of \( x_0 \) such that \( f(y_0) \in W \) implies \( f(V) \subset U \) for any \( f \in \mathcal{F} \).

Assume first \( x_0 \in X \). Given a neighbourhood \( U \) of \( x_0 \), let \( \epsilon > 0 \) be such that \( B_d(x_0, \epsilon) \), the open \( d \)-ball of center \( x_0 \) and radius \( \epsilon \), is contained in \( U \). Being \( \mathcal{F} \) equicontinuous, there is \( \delta > 0 \) so that

\[
d'(y, y_0) < \delta \Rightarrow d(f(y), f(y_0)) < \epsilon/2
\]

for all \( f \in \mathcal{F} \). We claim that \( V = B_d(y_0, \delta) \) and \( W = B_d(x_0, \epsilon/2) \) will do the job. Indeed, \( f(y_0) \in W \) means \( d(x_0, f(y_0)) < \epsilon/2 \), and so

\[
d(f(y), x_0) \leq d(f(y), f(y_0)) + d(f(y_0), x_0) < \epsilon/2 + \epsilon/2 = \epsilon,
\]

for every \( y \in V \), that is, \( f(V) \subset U \).

Now assume \( x_0 = \infty \), and let \( U = (X \setminus K) \cup \{\infty\} \) be a neighbourhood of \( x_0 \), where \( K \subset X \) is compact. Being \( X \) locally compact, we can find \( \epsilon > 0 \) such that \( H = \{x \in X | d(x, K) \leq \epsilon\} \) is still compact. Now, \( \mathcal{F} \) is equicontinuous; hence there is again \( \delta > 0 \) so that

\[
d'(y, y_0) < \delta \Rightarrow d(f(y), f(y_0)) < \epsilon/2
\]

for all \( f \in \mathcal{F} \). Then \( V = B_d(y_0, \delta) \) and \( W = (X \setminus H) \cup \{\infty\} \) will work. Indeed, \( f(y_0) \in W \) means \( d(f(y_0), K) > \epsilon \); so

\[
d(f(y), K) \geq d(f(y_0), K) - d(f(y), f(y_0)) > \epsilon - \epsilon/2 > 0,
\]

for every \( y \in V \), that is, \( f(V) \subset X \setminus K \subset U \). \( \square \)

And now we can prove
Theorem 1.3. A complex manifold \( X \) is hyperbolic if and only if \( \text{Hol}(\Delta, X) \) is relatively compact in \( C(\Delta, X^*) \).

Proof. Assume \( X \) hyperbolic; then \( k_X \) is a true distance inducing the manifold topology on \( X \) [B2]. Furthermore, if we consider \( \Delta \) endowed with the Poincaré distance then \( \text{Hol}(\Delta, X) \) is an equicontinuous family. If \( X \) is compact, \( k_X \) is complete, \( X \) is taut [Ki], and thus \( \text{Hol}(\Delta, X) \) is relatively compact in \( C(\Delta, X^*) \). If \( X \) is not compact, by Lemma 1.2, \( \text{Hol}(\Delta, X) \) is evenly continuous in \( C(\Delta, X^*) \); being \( X^* \) compact Hausdorff (and hence regular), Theorem 1.1 shows that \( \text{Hol}(\Delta, X) \) is relatively compact in \( C(\Delta, X^*) \).

Conversely, suppose \( X \) is not hyperbolic and take two distinct points \( z_0, w_0 \) such that \( k_X(z_0, w_0) = 0 \) (for the following argument, cf. [Ki]). Choose a coordinate neighbourhood \( U \) of \( z_0 \) relatively compact in \( X \) and biholomorphic to the unit ball \( B \) of \( \mathbb{C}^n \), where \( n \) is the complex dimension of \( X \), so that \( w_0 \notin \overline{U} \). Furthermore, fix another neighbourhood \( V \subset U \) of \( z_0 \).

We claim that for any \( \nu \in \mathbb{N} \) there is \( \varphi_\nu \in \text{Hol}(\Delta, X) \) such that \( \varphi_\nu(0) \in \overline{V} \) but \( \varphi_\nu(\Delta_1/\nu) \notin U \), where \( \Delta_r \) is the disk in \( \mathbb{C} \) centered in \( 0 \) of radius \( r \). This will yield the assertion: take \( \zeta_\nu \in \Delta_1/\nu \) such that \( \varphi_\nu(\zeta_\nu) \notin U \). If \( \text{Hol}(\Delta, X) \) were relatively compact in \( C(\Delta, X^*) \) then \( \{\varphi_\nu\} \) would have a subsequence \( \{\varphi_{\nu_k}\} \) converging to \( \varphi \in C(\Delta, X^*) \); but then \( \{\varphi_{\nu_k}(\zeta_{\nu_k})\} \) would converge to \( \varphi(0) \in \overline{V} \), which is impossible.

So we are left to prove the claim. Assume, by contradiction, that \( \nu \in \mathbb{N} \) is such that \( \varphi(0) \in \overline{V} \) implies \( \varphi(\Delta_1/\nu) \subset U \) for any \( \varphi \in \text{Hol}(\Delta, X) \). Choose a constant \( c > 0 \) such that \( \omega(0, \zeta) \geq c k_{\Delta_1/\nu}(0, \zeta) \) for all \( \zeta \in \Delta_1/(2\nu) \), and let \( \varepsilon = c k_U(z_0, \partial V) \). Since \( U \) is biholomorphic to \( B \), and hence hyperbolic, \( \varepsilon > 0 \).

Now, \( k_X(z_0, w_0) = 0 \) implies that we can find a sequence of points \( z_0, \ldots, z_m = w_0 \in X \) such that
\[
\sum_{j=1}^{m} \delta_X(z_{j-1}, z_j) < \frac{\varepsilon}{2},
\]
which in turn implies that we can find \( \zeta_j \in \Delta \) and \( \varphi_j \in \text{Hol}(\Delta, X) \) for \( j = 1, \ldots, m \) such that \( \varphi_j(0) = z_{j-1}, \varphi_j(\zeta_j) = z_j \) and \( \sum_{j=1}^{m} \omega(0, \zeta_j) < \varepsilon \). Let \( m_0 \leq m \) be the first integer such that \( \{\varphi_m(t\zeta_m)\mid t \in (0, 1)\} \not\subset V \). Adding enough points of the form \( t\zeta_j \) with \( t \in (0, 1) \) and \( j = 1, \ldots, m_0 \), we can assume that \( \zeta_j \in \Delta_1/(2\nu) \), \( \varphi_j(\zeta_j) \in V \) for \( j = 1, \ldots, m_0 - 1 \) and that \( \zeta_m \in \Delta_1 /(2\nu), \varphi_m(\zeta_m) \in \partial V \). Then
\[
\sum_{j=1}^{m} \omega(0, \zeta_j) \geq \sum_{j=1}^{m_0} \omega(0, \zeta_j) \geq c \sum_{j=1}^{m_0} k_{\Delta_1/\nu}(0, \zeta_j) \geq c \sum_{j=1}^{m_0} k_U(\varphi_j(0), \varphi_j(\zeta_j)) \geq c k_U(z_0, \varphi_m(\zeta_m)) \geq \varepsilon,
\]
which is a contradiction. □

References


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