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REPRESENTING POSITIVE HOMOLOGY CLASSES OF $\mathbf{C}P^2\#2\overline{\mathbf{C}P}^2$ AND $\mathbf{C}P^2\#3\overline{\mathbf{C}P}^2$

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ABSTRACT. Theorems of Donaldson are used to give a necessary and sufficient condition for a given second integral homology class ξ of $\mathbf{C}P^2\#n\overline{\mathbf{C}P}^2$ to be represented by a smoothly embedded 2-sphere (1) for $n = 2, 3$ and ξ positive (with self-intersection positive), and (2) for $n = 3$ and ξ characteristic. Case (2) is a consequence of a more general result on the characteristic embedding of 2-spheres into 4-manifolds, which result generalizes the theorem of Donaldson on spin 4-manifolds just as the result of Kervaire and Milnor on the characteristic embedding extends Rohlin's signature theorem.

1. INTRODUCTION

The problem of representing homology classes ξ of an almost definite 4-manifold M by smoothly embedded 2-spheres was completely solved only for $M = S^2 \times S^2$ by Kuga [K] and for $M = \mathbf{C}P^2\#\overline{\mathbf{C}P}^2$ by Lawson [L1] and independently by Luo [Lu]. As for $M = \mathbf{C}P^2\#2\overline{\mathbf{C}P}^2$, Lawson gave a complete answer to the problem restricted to characteristic homology classes [L1] and obtained partial results for ξ "positive," i.e., for ξ with self-intersection positive [L1, L2]. In this note, applying celebrated theorems of Donaldson [D], we wish to solve the problem (1) for $M = \mathbf{C}P^2\#2\overline{\mathbf{C}P}^2$ or $\mathbf{C}P^2\#3\overline{\mathbf{C}P}^2$, and ξ "positive", and (2) for $M = \mathbf{C}P^2\#3\overline{\mathbf{C}P}^2$ and ξ characteristic. While dealing with case (2), we wish also to make clear the relation between the theorem of Donaldson on spin 4-manifolds and the characteristic embedding of 2-spheres into 4-manifolds. We, therefore, will work throughout in the DIFF category, henceforth on the presupposition that all the manifolds and all the embeddings involved are understood to be smooth.

Given a closed connected oriented 4-manifold M and ξ, $\eta \in H_2(M; \mathbf{Z})$, we will denote by $b^+ = b^+(M)$ (resp. $b^- = b^-(M)$) the rank of the positive (resp. negative) part of the intersection form of M, by $\sigma = \sigma(M)$ the signature of M and by $\xi \cdot \eta$ the intersection number of ξ and η; and we will say that ξ is *representable by* S^2 if ξ can be represented by an embedded 2-sphere. For

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$M = \mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$, we will use the notation

$$(a\,;\, b_1\,,\, \dots\,,\, b_n) = au + b_1 v_1 + \cdots + b_n v_n \in H_2(M\,;\,\mathbf{Z})\,,$$

where a and b_i are integers and $\langle u\,;\, v_1\,,\,\dots\,,\, v_n\rangle$ is the natural basis with

$$u \cdot u = 1\,, \qquad u \cdot v_i = 0\,, \qquad v_i \cdot v_j = -\delta_{ij}.$$

Our main results are the following.

Theorem 1.1. *Let ξ be a class in $H_2(\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2\,;\,\mathbf{Z})$, $n = 2$ or 3, with $\xi \cdot \xi > 0$. Then ξ is representable by S^2 if and only if ξ is in the orbit of one of the classes $(k+1\,;\, k\,,\, 0(\,,\, 0))$, $(k+1\,;\, k\,,\, 1(\,,\, 0))$, $(2\,;\, 0\,,\, 0(\,,\, 0))$, under the action of the orthogonal group of the intersection form.*

Theorem 1.1 is a consequence of classical techniques of Wall [W2, W3] and a theorem of Donaldson [D, Theorem 1], which asserts that the intersection form of a closed connected oriented 4-manifold with $b^+ = 0$ must be equivalent over $\mathbf{Z}$ to the standard form $(-1) \oplus \cdots \oplus (-1)$. Note that Theorem 1.1 implies the known results on $\mathbf{C}P^2[T]$, $S^2 \times S^2[K]$, and $\mathbf{C}P^2 \# \overline{\mathbf{C}P}^2$ [L1, Lu]; see also 2.5. In §2, we will prove Theorem 1.1, giving an explicit procedure 2.3 to ascertain whether or not a given class ξ with $\xi \cdot \xi > 0$ is in such an orbit and thus representable by S^2.

Theorem 1.2. *Let $\xi \in H_2(\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2\,;\,\mathbf{Z})$ be characteristic.*

(1) *If $3 \le n \le 9$ and $\xi \cdot \xi = 1 - n = \sigma$, then ξ is representable by S^2.*
(2) *For $n = 3$, ξ is representable by S^2 if and only if $\xi \cdot \xi = -2 = \sigma$.*

Theorem 1.2 is a consequence of the techniques of Wall [W2, W3] and the following.

Theorem 1.3. *Let M be a closed connected oriented 4-manifold with the property $(*)$ that $H_1(M\,;\,\mathbf{Z})$ has no 2-torsion. Let $\xi \in H_2(M\,;\,\mathbf{Z})$ be a characteristic homology class representable by S^2.*

(1) *If $b^+ \le 3 \le b^-$, then $\xi \cdot \xi = \sigma(M) + 16m$ with $m \le [(b^- - 3)/16]$.*
(2) *If $b^+ < 3 < b^-$, then $\xi \cdot \xi = \sigma(M) + 16m$ with $m \le [(b^- - 4)/16]$.*
(3) *If $0 \le b^\pm \le 3$, then $\xi \cdot \xi = \sigma(M)$.*

Theorem 1.3 is a consequence of a second theorem of Donaldson [D, Theorem 2], which asserts that the intersection form of a closed connected oriented spin 4-manifold with the property $(*)$ above must be equivalent over $\mathbf{Z}$ to H or $H \oplus H$ according as $b^+ = 1$ or 2, where H is the standard hyperbolic form on $\mathbf{Z} \oplus \mathbf{Z}$. Note that Theorem 1.3 implies not only a result of Lawson [L1, Theorem 3] concerning 4-manifolds homeomorphic to $m\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$, $m = 1$ or 2, but also the fact that a homotopy $K3$ surface contains no characteristic 2-spheres with self-intersection positive. Furthermore, note that, although used in the proof of Theorem 1.3, Donaldson's Theorem 2 [D] can be regarded as a special case of Theorem 1.3, corresponding to the choice $\xi = 0$: in other words, Theorem 1.3 is to Donaldson's Theorem 2 [D] what Kervaire and Milnor's congruence [KM, Theorem 1] is to Rohlin's signature theorem [R]. In §3, we will prove Theorems 1.2 and 1.3, working in connection with the $11/8$ conjecture, which might restrict the existence of spin 4-manifolds (cf. [Mt, §1; FM, §4]).

2. Timelike 2-spheres in almost definite 4-manifolds

This section is devoted to the proof of Theorem 1.1. Let us begin by preparing a key lemma, which is obtained by modifying the proof of a theorem of Kuga [K, Theorem 2].

Lemma 2.1. *Let M be a closed connected oriented 4-manifold with $b^+ = 1$ and $b^- \geq 1$. Suppose that $\xi \in H_2(M; \mathbf{Z})$ satisfies*

(i) $\xi \cdot \xi \geq 5$;

(ii) *there is no class $\eta \in H_2(M; \mathbf{Z})$ such that $\xi \cdot \eta = 1$ and $\eta \cdot \eta = 0$.*

Then ξ is not representable by S^2.

Proof. Suppose that ξ is represented by an embedded 2-sphere S in M. "Blow up" $n := \xi \cdot \xi - 1$ (≥ 4) distinct points of S, and then "blow down" the resulting "exceptional curve" of self-intersection $+1$, so as to construct a closed connected oriented negative-definite 4-manifold N; namely,

$$(M', S') := (M, S)\#n(\overline{\mathbf{C}P}^2, \overline{\mathbf{C}P}^1) \cong (N, \phi)\#(\mathbf{C}P^2, \mathbf{C}P^1).$$

Donaldson's Theorem 1 [D] implies that the intersection form of N must be standard, or equivalently,

$$\text{(A)} \qquad \#\{\alpha \in \operatorname{Fr} H_2(N);\, \alpha \cdot \alpha = -1\} = 2(b^- + n),$$

where $\operatorname{Fr} H_2(N)$ means the free part of $H_2(N; \mathbf{Z})$ under an implicitly fixed splitting of $H_2(N; \mathbf{Z})$. Letting $\gamma \in H_2(M'; \mathbf{Z})$ be the class represented by the 2-sphere S' in M', we see that (A) is equivalent to

$$\text{(B)} \qquad \#\{\alpha \in \operatorname{Fr} H_2(M');\, \gamma \cdot \alpha = 0,\ \alpha \cdot \alpha = -1\} = 2(b^- + n).$$

Let $\zeta_i = [\overline{\mathbf{C}P}^1]$ $(1 \leq i \leq n)$ denote the generator of $H_2(\overline{\mathbf{C}P}^2; \mathbf{Z})$, and express γ and α in (B) as

$$\gamma = \xi + \zeta_1 + \cdots + \zeta_n, \qquad \alpha = \eta + z_1\zeta_1 + \cdots + z_n\zeta_n,$$

where $\eta \in \operatorname{Fr} H_2(M)$ and $z_i \in \mathbf{Z}$ $(1 \leq i \leq n)$. (B) then is equivalent to

$$\text{(C)} \qquad \#\{(\eta; z_1, \dots, z_n) \in \operatorname{Fr} H_2(M) \oplus \mathbf{Z}^n;\, (D)\} = 2(b^- + n),$$

where (D) is the following system of Diophantine equations:

$$\text{(D)} \qquad \xi \cdot \eta - z_1 - \cdots - z_n = 0, \qquad \eta \cdot \eta - z_1^2 - \cdots - z_n^2 = -1.$$

Since $n = \xi \cdot \xi - 1 \geq 4$ by hypothesis (i), it cannot happen that all the solutions to (D) are of form $(\eta; 0, \dots, 0)$. Hence, there exists such an η in $\operatorname{Fr} H_2(M)$ as satisfies

$$\text{(E1)} \qquad \xi \cdot \eta = z_1 + \cdots + z_r,$$

$$\text{(E2)} \qquad \eta \cdot \eta = z_1^2 + \cdots + z_r^2 - 1,$$

$$\text{(E3)} \qquad z_i \neq 0 \qquad (1 \leq i \leq r),$$

$$\text{(E4)} \qquad 1 \leq r \leq \xi \cdot \xi - 1.$$

Now, we claim

$$(\mathrm{E3}') \qquad z_i = \pm 1 \qquad (1 \le i \le r).$$

In fact, since ξ is "timelike" $(\xi \cdot \xi > 0)$, (E1) and the "reverse Cauchy-Schwarz inequality" for Lorentzian spaces give

$$(\xi \cdot \xi)(\eta \cdot \eta) \le (\xi \cdot \eta)^2 = (z_1 + \cdots + z_r)^2,$$

where equality holds if and only if η is parallel to ξ in $\mathrm{Fr}\, H_2(M)$. Then it follows from (E2) that

$$(\xi \cdot \xi)(\eta \cdot \eta) \le (z_1 + \cdots + z_r)^2 \le r(z_1^2 + \cdots + z_r^2) = r(\eta \cdot \eta + 1).$$

Thus (E2), (E3), and (E4) imply

$$0 \le \eta \cdot \eta \le (\xi \cdot \xi - r)(\eta \cdot \eta) \le r,$$
$$r \le z_1^2 + \cdots + z_r^2 = \eta \cdot \eta + 1 \le r + 1,$$

hence (E3$'$).

Setting $s = \#\{i;\, z_i = -1\}$, we further claim

$$(\mathrm{E3}'') \qquad s = 0 \quad \text{or} \quad s = r.$$

In fact, (E1) and (E2) become

$$(\mathrm{E1}') \qquad \xi \cdot \eta = r - 2s,$$

$$(\mathrm{E2}') \qquad \eta \cdot \eta = r - 1,$$

and (E1$'$), (E2$'$), (E4), and the "reverse Cauchy-Schwarz inequality" imply

$$\begin{aligned} 0 &\le (r - 2s)^2 - (\xi \cdot \xi)(r - 1) =: \Delta \\ &\le (r - 2s)^2 - (r + 1)(r - 1) = 1 - 4s(r - s) \le 1, \end{aligned}$$

hence (E3$''$).

By changing the sign of η if necessary, we assume that $s = 0$.

From the above evaluation of Δ, we also obtain either of the following:

$$(\mathrm{F1}) \qquad \Delta = r^2 - (\xi \cdot \xi) r + (\xi \cdot \xi) = 0,$$

$$(\mathrm{F2}) \qquad \Delta - 1 = r^2 - (\xi \cdot \xi) r + (\xi \cdot \xi - 1) = 0.$$

Case (F1) is impossible to occur, since its discriminant, $(\xi \cdot \xi)^2 - 4(\xi \cdot \xi)$, cannot be the square of any integer because of hypothesis (i). Thus we have

$$(\mathrm{F2}') \qquad r = 1 \quad \text{or} \quad r = \xi \cdot \xi - 1.$$

However, (F2$'$) contradicts hypothesis (ii), since the simultaneous equations

$$\xi \cdot \eta = \xi \cdot \xi - 1, \qquad \eta \cdot \eta = \xi \cdot \xi - 2$$

are equivalent to

$$\xi \cdot (\xi - \eta) = 1, \qquad (\xi - \eta) \cdot (\xi - \eta) = 0.$$

This contradiction completes the proof. $\square$

2.2. *Facts.* Let M be $\mathbf{C}P^2 \# n \overline{\mathbf{C}P}^2$ with $n \ge 2$, and $O(M)$ the orthogonal group of the intersection form of M. Recall the following facts: cf. [W2, 1.5, 1.6; W3, Lemma 2 and Corollary to Theorem 2].

(1) Orientation-preserving diffeomorphisms of M realize the following reflections in $O(M)$: for $\xi = (b_0; b_1, \dots, b_n) \in H_2(M; \mathbf{Z})$,

$$R_i: \xi \to (b_0; b_1, \dots, -b_i, \dots, b_n), \qquad 0 \le i \le n.$$
$$R_{jk}: \xi \to (b_0; b_1, \dots, b_k, \dots, b_j, \dots, b_n), \qquad 1 \le j < k \le n.$$
$$R: \xi \to \xi + \begin{cases} 2(b_0 - b_1 - b_2)(1; 1, 1), & n = 2, \\ (b_0 - b_1 - b_2 - b_3)(1; 1, 1, 1, 0, \dots, 0), & n \ge 3. \end{cases}$$

(2) Reflections R_i, R_{jk}, and R above generate $O(M)$ if $2 \le n \le 9$.

Criterion 2.3. *Let ξ be a class in $H_2(\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2; \mathbf{Z})$, $n \ge 2$.*

(1) *If $\xi \cdot \xi > 0$, then ξ is transformed, with reflections R_i, R_{jk}, and R of* 2.2(1), *into $(a; b_1, \dots, b_n)$ such that*

$$a > b_1 \ge \cdots \ge b_n \ge 0, \qquad a \ge \begin{cases} b_1 + b_2, & n = 2, \\ b_1 + b_2 + b_3, & n \ge 3. \end{cases}$$

(2) *If $\xi \cdot \xi \ge 5$ and $(a; b_1, \dots, b_n)$ in* (1) *satisfies*

$$(a-1)^2 > b_1^2 + \cdots + b_n^2,$$

then ξ is not representable by S^2.

Proof. (1) The case for $n = 2$ is nothing but 2.3 of [W2]. Set $n \ge 3$ and $\xi = (a; b_1, \dots, b_n)$. Letting R_i and R_{jk} act if necessary, assume that

$$a > b_1 \ge \cdots \ge b_n \ge 0.$$

Then observe that $R\xi = (2a - b_1 - b_2 - b_3; \dots)$ and that

$$a < b_1 + b_2 + b_3 \quad \text{implies} \quad |2a - b_1 - b_2 - b_3| < a,$$

thus obtaining the assertion.

(2) By virtue of Lemma 2.1, it is enough to show that there exists no class $\eta \in H_2(\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2; \mathbf{Z})$ such that $\xi \cdot \eta = 1$ and $\eta \cdot \eta = 0$. Suppose that η is such a class. Without loss of generality, assume that

$$\xi = (a; b_1, \dots, b_n) = (a; b), \qquad |a| - |b| > 1, \ b \in \mathbf{Z}^n.$$

Note that for $\eta = (a'; b') \in \mathbf{Z} \times \mathbf{Z}^n$, $|a'| = |b'| \ge 1$. Then, the "ordinary" Cauchy-Schwarz inequality for $\mathbf{Z}^n$ gives

$$1 = |\xi \cdot \eta| \ge (|a| - |b|)|a'| > 1,$$

a contradiction. □

2.4. *Proof of Theorem* 1.1. Taking a connected sum with $\overline{\mathbf{C}P}^2$ if $n = 2$, assume that $n = 3$. As in 2.3(1), transform a given class ξ, with R_i, R_{jk}, and R, into $(a; b, c, d)$ such that

$$a > b \ge c \ge d \ge 0, \qquad a \ge b + c + d.$$

Then, it is easy to verify the following:

(1) $a - b \ge 2$ implies $(a-1)^2 > b^2 + c^2 + d^2$;
(2) $a - b \ge 2$ and $\xi \cdot \xi \le 4$ imply $(a; b, c, d) = (2; 0, 0, 0)$.

Hence, Theorem 1.1 follows immediately from (1), (2), 2.3(2), 2.2, and the fact that the classes $(k+1; k, 0, 0)$, $(k+1; k, 1, 0)$, and $(2; 0, 0, 0)$ are representable by S^2; cf. [W3, §1; L1, §2]. □

2.5. *Remarks.* (1) The problem of representing homology classes by embedded 2-spheres for a closed connected oriented 4-manifold M with $b^+ = 1$ was completely solved only for $M = \mathbf{C}P^2[T]$, $S^2 \times S^2[K]$, and $\mathbf{C}P^2\#\overline{\mathbf{C}P}^2$ [L1, Lu]. Note that those results are now corollaries to Theorem 1.1: reverse the orientation of M if necessary, blow up a point or two of M, and apply Theorem 1.1.

(2) Let M be either $S^2 \times S^2$ or $\mathbf{C}P^2\#\overline{\mathbf{C}P}^2$. Recall [W3, Lemma 1] that the latter can be considered as the twisted 2-sphere bundle over S^2, $S^2 \tilde{\times} S^2$, while the former is the other trivial one. It is interesting to observe from this point of view that $\xi \in H_2(M; \mathbf{Z})$ is representable by S^2 if and only if ξ corresponds to one of the following three cases: fiber(s), fiber(s) with one cross section, two cross sections for $M = S^2 \tilde{\times} S^2$.

(3) Lawson treated only the following classes in $H_2(\mathbf{C}P^2\#2\overline{\mathbf{C}P}^2; \mathbf{Z})$:

$$(2k+3; 2k+1, 2m), \quad k \geq m \geq 1, \quad \text{square } \geq 5 \quad \text{[L1, Theorem 4]};$$
$$(2k+2; 2k, 2m+1), \quad k > m \geq 1, \quad \text{square } \geq 11 \quad \text{[L2, Proposition 5]}.$$

Note that such a class is reduced with our procedure 2.3(1) to one of

$$(2k'+3; 2k'+1, 0), \qquad (2k'+3; 2k'+1, 2), \qquad (2k'+2; 2k', 1);$$

and therefore is forbidden by Theorem 1.1 to be represented by an embedded 2-sphere.

(4) From a result of Wall [W3, p. 138] and a result of Hirai [H, Theorem 7], it follows that if $\xi \in H_2(\mathbf{C}P^2\#2\overline{\mathbf{C}P}^2; \mathbf{Z})$ is ordinary with $-12 < \xi \cdot \xi < 8$, then ξ is representable by S^2. According to Li [Li, Theorem 6], it turns out that if $\xi \in H_2(\mathbf{C}P^2\#n\overline{\mathbf{C}P}^2; \mathbf{Z})$, $2 \leq n \leq 9$, is ordinary with $\xi \cdot \xi = 0$, then ξ is representable by S^2. On the other hand, note that for any integer r, there is a class $\xi \in H_2(\mathbf{C}P^2\#n\overline{\mathbf{C}P}^2; \mathbf{Z})$, $n \geq 2$, representable by S^2 with $\xi \cdot \xi = r$: e.g., $\xi = (k+1; k, 0, 0, \ldots, 0)$ or $(k+1; k, 1, 0, \ldots, 0)$.

3. Characteristic 2-spheres in 4-manifolds

This section is devoted to the proofs of Theorems 1.2 and 1.3. Let us begin by introducing an extended version of the $11/8$ conjecture (cf. [Mt, §1; FM, §4]).

Conjecture 3.1. *The signature of a closed oriented spin 4-manifold M must satisfy the inequality $b_2(M) \geq (11/8)|\sigma(M)|$.*

3.2. *Notation.* For an integer $r > 0$, let $\mathrm{CONJ}(r, P)$ denote the proposition that Conjecture 3.1 is true of any closed connected oriented spin 4-manifold M with $b^+ < r$ and with a particular property P on $H_1(M; \mathbf{Z})$.

3.3. *Remarks.* (1) By Rohlin's signature theorem [R], the inequality $b_2 \geq (11/8)|\sigma|$ in Conjecture 3.1 is equivalent to

$$b^- = b^+ + 16m, \quad m \in \mathbf{Z}, \quad -[b^+/19] \leq m \leq [b^+/3].$$

(2) Donaldson's Theorem 2 [D] is equivalent to $\mathrm{CONJ}(3, (*))$, where $(*)$ is the property that $H_1(M; \mathbf{Z})$ has no 2-torsion.

Lemma 3.4. *Let M be a closed connected oriented 4-manifold with $b^+ \leq r$ and with a particular property P on $H_1(M; \mathbf{Z})$, and $\xi \in H_2(M; \mathbf{Z})$ a characteristic homology class representable by S^2. Then* CONJ(r, P) *implies $\xi \cdot \xi = \sigma(M) + 16m$ for some integer m such that*

(1) *if $b^+ = r$, then $m \leq \max\{[(r-1)/3], [(b^- - r)/16]\}$;*
(2) *if $b^+ < r$, then $m \leq \max\{[b^+/3], [(b^- - b^+ - 2)/16]\}$.*

Proof. Note that Kervaire and Milnor's generalization [KM, Theorem 1] of Rohlin's signature theorem implies $\xi \cdot \xi = \sigma(M) + 16m$ for some integer m.

(1) Suppose that m does not satisfy the above inequality: namely,

$$m > \max\{[(r-1)/3], [(b^- - r)/16]\}.$$

Then $\xi \cdot \xi = r - b^- + 16m > 0$. Let a 2-sphere $S \subset M$ represent ξ, "blow up" $\xi \cdot \xi - 1$ (≥ 0) distinct point(s) of S, and "blow down" the resulting "exceptional curve" of self-intersection $+1$: namely,

$$(M, S)\#(\xi \cdot \xi - 1)(\overline{\mathbf{C}P}^2, \overline{\mathbf{C}P}^1) \cong (N, \phi)\#(\mathbf{C}P^2, \mathbf{C}P^1),$$

thereby constructing a closed connected oriented spin 4-manifold N with the property P on $H_1(N; \mathbf{Z})$ and with $b^+(N) = r - 1$, $b^-(N) = b^+(N) + 16m$, and $m > [b^+(N)/3]$. By 3.3(1), the existence of such an N is contradictory to CONJ(r, P).

(2) The proof is similar. Suppose not. Let a 2-sphere $S \subset M$ represent ξ, "blow up" $\xi \cdot \xi + 1$ (≥ 0) distinct point(s) of S, and "blow down" the resulting "exceptional curve" of self-intersection -1, thereby producing a spin 4-manifold whose existence is contradictory to CONJ(r, P). □

3.5. *Proof of Theorem* 1.3. Let $S \subset M$ be a 2-sphere representing ξ.

(1) immediately follows from 3.3(2) and Lemma 3.4(1).

(2) By taking the connected sum $(M, S)\#(2 - b^+)(\mathbf{C}P^2, \mathbf{C}P^1)$, reduce the case to $b^+ = 2$, to obtain the assertion through 3.3(2) and Lemma 3.4(2).

(3) First consider the case $b^+ = b^- = 3$. Note that (1) implies $\xi \cdot \xi \leq 0$. Reversing the orientation of M, observe that $\xi \cdot \xi = 0 = \sigma(M)$. Next suppose either $b^+ < 3$ or $b^- < 3$. Take the connected sum

$$(M, S)\#(3 - b^+)(\mathbf{C}P^2, \mathbf{C}P^1)\#(3 - b^-)(\overline{\mathbf{C}P}^2, \overline{\mathbf{C}P}^1),$$

thereby obtaining $\xi \cdot \xi - b^+ + b^- = 0$, i.e., $\xi \cdot \xi = b^+ - b^- = \sigma(M)$. □

3.6. *Proof of Theorem* 1.2. As for (2), the "if" (resp. "only if") part is contained in (1) (resp. Theorem 1.3(3)). Thus, in virtue of 2.2, it is enough to show that any characteristic homology class

$$\xi = (a; b_1, \ldots, b_n) \in H_2(\mathbf{C}P^2\#n\overline{\mathbf{C}P}^2; \mathbf{Z}), \quad 3 \leq n \leq 9, \ \xi \cdot \xi = 1 - n,$$

is transformed, with reflections R_i, R_{jk}, and R, into $(1; 1, \ldots, 1)$, which is representable by S^2. First note that a and b_i are odd, and that

$$a^2 - b_1^2 - \cdots - b_n^2 = 1 - n.$$

Next assume, by letting R_i and R_{jk} act if necessary, that

$$a \geq b_1 \geq \cdots \geq b_n \geq 1.$$

Then, it is easy to verify that if $a > 1$, then

$$a < b_1 + b_2 + b_3 < 3a;$$

and hence that if $a > 1$, then

$$|2a - b_1 - b_2 - b_3| < a.$$

Therefore, reflection R, which transforms ξ into $(2a - b_1 - b_2 - b_3; \dots)$, strictly reduces a whenever $a > 1$. This completes the proof. □

3.7. *Remarks.* (1) Because of Theorem 1.3(3), Theorem 1.2(2) is generalized as follows. Let M be $m\mathbf{C}P^2\#n\overline{\mathbf{C}P}^2$ or $r(S^2 \times S^2)$, $0 \leq m, n, r \leq 3$: when $m = n = 0$ or $r = 0$, M is understood to be S^4. Suppose that $\xi \in H_2(M; \mathbf{Z})$ is characteristic. Then ξ is representable by S^2 if and only if $\xi \cdot \xi = \sigma(M)$. Confer [K, Theorem 1; L1, Theorems 1, 2; Lu, Theorem; H, Theorems 1, 2, 3; Li, §2].

(2) There are 4-manifolds M that possess characteristic homology classes that are representable by S^2 but have self-intersection different from the signature of M. For example, let M be $19n(\mathbf{C}P^2\#\overline{\mathbf{C}P}^2)$, $n \geq 1$. Recall [Mn, Theorem 2.4] that a $K3$ surface K is almost completely decomposable, i.e., that $K\#\mathbf{C}P^2$ is diffeomorphic to $4\mathbf{C}P^2\#19\overline{\mathbf{C}P}^2$. Hence, M is diffeomorphic to $n(K\#16\mathbf{C}P^2)$. It follows from this fact, together with Theorem 4 of [W1] and Corollary to Theorem 2 of [W3], that if ξ is a primitive characteristic homology class in $H_2(M; \mathbf{Z})$ with $\xi \cdot \xi = 16i$, $|i| \leq n$, then ξ is representable by S^2 (cf. [W4, (4.7)]).

We conclude by stating a corollary to Theorem 1.3, which contains a result of Lawson [L1, Theorem 5] (cf. [Li, §3]).

Corollary 3.8. *Let M be a closed 1-connected oriented 4-manifold.*

(1) *If $b^{\pm} \leq 2$, then the normal Euler number of a characteristic embedding of $\mathbf{R}P^2$ in M is either of $\sigma(M) \pm 2$, both of which can be realized for $M = m\mathbf{C}P^2\#n\overline{\mathbf{C}P}^2$ or $r(S^2 \times S^2)$, $0 \leq m, n, r \leq 2$.*

(2) *If $b^{\pm} \leq 1$, then the normal Euler number of a characteristic embedding of Klein's bottle Kl in M is one of $\sigma(M)$ and $\sigma(M) \pm 4$, all of which can be realized for $M = m\mathbf{C}P^2\#n\overline{\mathbf{C}P}^2$ or $r(S^2 \times S^2)$, $0 \leq m, n, r \leq 1$.*

Proof. (1) (resp.(2)) immediately follows from Theorem 1.3(3), 3.7(1) and Rohlin-Lawson's "Connecting Lemma" of [L1,§5], which asserts that if ν is the normal Euler number of a characteristic embedding of $\mathbf{R}P^2$ (resp. Kl) in a closed 1-connected oriented 4-manifold M, then there is an embedding of S^2 into $M\#(S^2 \times S^2)$ (resp. $M\#2(S^2 \times S^2)$), representing a characteristic homology class ξ with $\xi \cdot \xi$ equal to either of $\nu \pm 2$ (resp. one of ν and $\nu \pm 4$). □

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References

[D] S. K. Donaldson, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Differential Geom. **26** (1987), 379–428.

[FM] R. Friedman and J. W. Morgan, *Algebraic surfaces and* 4-*manifolds: some conjectures and speculations*, Bull. Amer. Math. Soc. (N.S.) **18** (1988), 1–19.

[H] Y. Hirai, *Representing homology classes of connected sums of* 2-*sphere bundles over* 2-*spheres*, Kobe J. Math. **6** (1989), 233–240.

[KM] M. Kervaire and J. W. Milnor, *On* 2-*spheres in* 4-*manifolds*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 1651–1657.

[K] K. Kuga, *Representing homology classes of* $S^2 \times S^2$, Topology **23** (1984), 133–137.

[L1] T. Lawson, *Representing homology classes of almost definite* 4-*manifolds*, Michigan Math. J. **34** (1987), 85–91.

[L2] ——, *Compactness results for orbifold instantons*, Math. Z. **200** (1988), 123–140.

[Li] B. H. Li, *Embeddings of surfaces in* 4-*manifolds*, preprint, 1990.

[Lu] F. Luo, *Representing homology classes of* $\mathbf{C}P^2\#\overline{\mathbf{C}P}^2$, Pacific J. Math. **133** (1988), 137–140.

[Mn] R. Mandelbaum, *Four-dimensional topology: an introduction*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), 1–159.

[Mt] Y. Matsumoto, *On the bounding genus of homology* 3-*spheres*, J. Fac. Sci. Univ. Tokyo **29** (1982), 287–318.

[R] V. A. Rohlin, *New results in the theory of* 4-*dimensional manifolds*, Dokl. Akad. Nauk SSSR **84** (1952), 221–224. (Russian)

[T] A. G. Tristram, *Some cobordism invariants for links*, Proc. Cambridge Philos. Soc. **66** (1969), 251–264.

[W1] C. T. C. Wall, *On the orthogonal groups of unimodular quadratic forms*, Math. Ann. **147** (1962), 328–338.

[W2] ——, *On the orthogonal groups of unimodular quadratic forms*. II, J. Reine Angew. Math. **213** (1963), 122–136.

[W3] ——, *Diffeomorphisms of* 4-*manifolds*, J. London Math. Soc. (2) **39** (1964), 131–140.

[W4] ——, *On simply-connected* 4-*manifolds*, J. London Math. Soc. (2) **39** (1964), 141–149.

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