REPRESENTING POSITIVE HOMOLOGY CLASSES
OF $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ AND $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$

KAZUNORI KIKUCHI

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Abstract. Theorems of Donaldson are used to give a necessary and sufficient condition for a given second integral homology class $\zeta$ of $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ to be represented by a smoothly embedded 2-sphere (1) for $n = 2, 3$ and $\zeta$ positive (with self-intersection positive), and (2) for $n = 3$ and $\zeta$ characteristic. Case (2) is a consequence of a more general result on the characteristic embedding of 2-spheres into 4-manifolds, which result generalizes the theorem of Donaldson on spin 4-manifolds just as the result of Kervaire and Milnor on the characteristic embedding extends Rohlin’s signature theorem.

1. Introduction

The problem of representing homology classes $\zeta$ of an almost definite 4-manifold $M$ by smoothly embedded 2-spheres was completely solved only for $M = S^2 \times S^2$ by Kuga [K] and for $M = \mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ by Lawson [L1] and independently by Luo [Lu]. As for $M = \mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$, Lawson gave a complete answer to the problem restricted to characteristic homology classes [L1] and obtained partial results for $\zeta$ “positive,” i.e., for $\zeta$ with self-intersection positive [L1, L2]. In this note, applying celebrated theorems of Donaldson [D], we wish to solve the problem (1) for $M = \mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ or $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, and $\zeta$ “positive”, and (2) for $M = \mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ and $\zeta$ characteristic. While dealing with case (2), we wish also to make clear the relation between the theorem of Donaldson on spin 4-manifolds and the characteristic embedding of 2-spheres into 4-manifolds. We, therefore, will work throughout in the DIFF category, henceforth on the presupposition that all the manifolds and all the embeddings involved are understood to be smooth.

Given a closed connected oriented 4-manifold $M$ and $\zeta, \eta \in H_2(M; \mathbb{Z})$, we will denote by $b^+ = b^+(M)$ (resp. $b^- = b^-(M)$) the rank of the positive (resp. negative) part of the intersection form of $M$, by $\sigma = \sigma(M)$ the signature of $M$ and by $\zeta \cdot \eta$ the intersection number of $\zeta$ and $\eta$ and we will say that $\zeta$ is representable by $S^2$ if $\zeta$ can be represented by an embedded 2-sphere. For
$M = \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, we will use the notation

$$(a; b_1, \ldots, b_n) = au + b_1v_1 + \cdots + b_nv_n \in H_2(M; \mathbb{Z}),$$

where $a$ and $b_i$ are integers and $\{u; v_1, \ldots, v_n\}$ is the natural basis with

$$u \cdot u = 1, \quad u \cdot v_i = 0, \quad v_i \cdot v_j = -\delta_{ij}.$$

Our main results are the following.

**Theorem 1.1.** Let $\xi$ be a class in $H_2(\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}; \mathbb{Z})$, $n = 2$ or 3, with $\xi \cdot \xi > 0$. Then $\xi$ is representable by $S^2$ if and only if $\xi$ is in the orbit of one of the classes $(k+1; k, 0, 0, 0), (k+1; k, 1, 0, 0), (2; 0, 0, 0, 0)$, under the action of the orthogonal group of the intersection form.

Theorem 1.1 is a consequence of classical techniques of Wall [W2, W3] and a theorem of Donaldson [D, Theorem 1], which asserts that the intersection form of a closed connected oriented 4-manifold with $b^+ = 0$ must be equivalent over $\mathbb{Z}$ to the standard form $(-1) \oplus \cdots \oplus (-1)$. Note that Theorem 1.1 implies the known results on $\mathbb{C}P^2[T], S^2 \times S^2[K], \text{ and } \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ [L1, Lu]; see also 2.5.

In §2, we will prove Theorem 1.1, giving an explicit procedure to ascertain whether or not a given class $\xi$ with $\xi \cdot \xi > 0$ is in such an orbit and thus representable by $S^2$.

**Theorem 1.2.** Let $\xi \in H_2(\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}; \mathbb{Z})$ be characteristic.

1. If $3 \leq n \leq 9$ and $\xi \cdot \xi = 1 - n = \sigma$, then $\xi$ is representable by $S^2$.
2. For $n = 3$, $\xi$ is representable by $S^2$ if and only if $\xi \cdot \xi = -2 = \sigma$.

Theorem 1.2 is a consequence of the techniques of Wall [W2, W3] and the following.

**Theorem 1.3.** Let $M$ be a closed connected oriented 4-manifold with the property (*) that $H_1(M; \mathbb{Z})$ has no 2-torsion. Let $\xi \in H_2(M; \mathbb{Z})$ be a characteristic homology class representable by $S^2$.

1. If $b^+ \leq 3 \leq b^-$, then $\xi \cdot \xi = \sigma(M) + 16m$ with $m \leq [(b^- - 3)/16]$.
2. If $b^+ < 3 < b^-$, then $\xi \cdot \xi = \sigma(M) + 16m$ with $m \leq [(b^- - 4)/16]$.
3. If $0 \leq b^\pm \leq 3$, then $\xi \cdot \xi = \sigma(M)$.

Theorem 1.3 is a consequence of a second theorem of Donaldson [D, Theorem 2], which asserts that the intersection form of a closed connected oriented spin 4-manifold with the property (*) above must be equivalent over $\mathbb{Z}$ to $H$ or $H \oplus H$ according as $b^+ = 1$ or 2, where $H$ is the standard hyperbolic form on $\mathbb{Z} \oplus \mathbb{Z}$. Note that Theorem 1.3 implies not only a result of Lawson [L1, Theorem 3] concerning 4-manifolds homeomorphic to $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, $m = 1$ or 2, but also the fact that a homotopy $K3$ surface contains no characteristic 2-spheres with self-intersection positive. Furthermore, note that, although used in the proof of Theorem 1.3, Donaldson’s Theorem 2 [D] can be regarded as a special case of Theorem 1.3, corresponding to the choice $\xi = 0$; in other words, Theorem 1.3 is to Donaldson’s Theorem 2 [D] what Kervaire and Milnor’s congruence [KM, Theorem 1] is to Rohlin’s signature theorem [R]. In §3, we will
prove Theorems 1.2 and 1.3, working in connection with the 11/8 conjecture, which might restrict the existence of spin 4-manifolds (cf. [Mt, §1; FM, §4]).

2. Timelike 2-spheres in almost definite 4-manifolds

This section is devoted to the proof of Theorem 1.1. Let us begin by preparing a key lemma, which is obtained by modifying the proof of a theorem of Kuga [K, Theorem 2].

**Lemma 2.1.** Let $M$ be a closed connected oriented 4-manifold with $b^+ = 1$ and $b^- \geq 1$. Suppose that $\xi \in H_2(M; \mathbb{Z})$ satisfies

(i) $\xi \cdot \xi \geq 5$;

(ii) there is no class $\eta \in H_2(M; \mathbb{Z})$ such that $\xi \cdot \eta = 1$ and $\eta \cdot \eta = 0$.

Then $\xi$ is not representable by $S^2$.

**Proof.** Suppose that $\xi$ is represented by an embedded 2-sphere $S$ in $M$. "Blow up" $n := \xi \cdot \xi - 1 (\geq 4)$ distinct points of $S$, and then "blow down" the resulting "exceptional curve" of self-intersection $+1$, so as to construct a closed connected oriented negative-definite 4-manifold $N$; namely,

$$(M', S') := (M, S) \# n(CP^2, CP^1) \# (CP^2, CP^1).$$

Donaldson's Theorem 1 [D] implies that the intersection form of $N$ must be standard, or equivalently,

$$(A) \quad \# \{\alpha \in \text{Fr} H_2(N); \alpha \cdot \alpha = -1\} = 2(b^- + n),$$

where $\text{Fr} H_2(N)$ means the free part of $H_2(N; \mathbb{Z})$ under an implicitly fixed splitting of $H_2(N; \mathbb{Z})$. Letting $\gamma \in H_2(M'; \mathbb{Z})$ be the class represented by the 2-sphere $S'$ in $M'$, we see that (A) is equivalent to

$$(B) \quad \# \{\alpha \in \text{Fr} H_2(M'); \gamma - \alpha = 0, \alpha \cdot \alpha = -1\} = 2(b^- + n).$$

Let $\zeta_i = [CP^1]$ ($1 \leq i \leq n$) denote the generator of $H_2(CP^2; \mathbb{Z})$, and express $\gamma$ and $\alpha$ in (B) as

$$\gamma = \xi + \zeta_1 + \cdots + \zeta_n, \quad \alpha = \eta + z_1 \zeta_1 + \cdots + z_n \zeta_n,$$

where $\eta \in \text{Fr} H_2(M)$ and $z_i \in \mathbb{Z}$ ($1 \leq i \leq n$). (B) then is equivalent to

$$(C) \quad \# \{(\eta; z_1, \ldots, z_n) \in \text{Fr} H_2(M) \oplus \mathbb{Z}^n; (D)\} = 2(b^- + n),$$

where (D) is the following system of Diophantine equations:

$$(D) \quad \xi \cdot \eta - z_1 - \cdots - z_n = 0, \quad \eta \cdot \eta - z_1^2 - \cdots - z_n^2 = -1.$$

Since $n = \xi \cdot \xi - 1 \geq 4$ by hypothesis (i), it cannot happen that all the solutions to (D) are of form $(\eta; 0, \ldots, 0)$. Hence, there exists such an $\eta$ in $\text{Fr} H_2(M)$ as satisfies

$$(E1) \quad \xi \cdot \eta = z_1 + \cdots + z_r,$$

$$(E2) \quad \eta \cdot \eta = z_1^2 + \cdots + z_r^2 - 1,$$

$$(E3) \quad z_i \neq 0 \quad (1 \leq i \leq r),$$

$$(E4) \quad 1 \leq r \leq \xi \cdot \xi - 1.$$
Now, we claim

\[(E3')\quad z_i = \pm 1 \quad (1 \leq i \leq r).\]

In fact, since \(\xi\) is "timelike" \((\xi \cdot \xi > 0)\), \((E1)\) and the "reverse Cauchy-Schwarz inequality" for Lorentzian spaces give

\[(\xi \cdot \xi) (\eta \cdot \eta) \leq (\xi \cdot \eta)^2 = (z_1 + \cdots + z_r)^2,\]

where equality holds if and only if \(\eta\) is parallel to \(\xi\) in \(Fr H_2(M)\). Then it follows from \((E2)\) that

\[(\xi \cdot \xi) (\eta \cdot \eta) \leq (z_1 + \cdots + z_r)^2 \leq r(z_1^2 + \cdots + z_r^2) = r(\eta \cdot \eta + 1).\]

Thus \((E2), (E3),\) and \((E4)\) imply

\[
0 \leq \eta \cdot \eta \leq (\xi \cdot \xi - r)(\eta \cdot \eta) \leq r, \\
\eta \cdot \eta + 1 \leq r + 1,
\]

hence \((E3')\).

Setting \(s = \# \{ i ; z_i = -1 \}\), we further claim

\[(E3'')\quad s = 0 \quad \text{or} \quad s = r.\]

In fact, \((E1)\) and \((E2)\) become

\[(E1')\quad \xi \cdot \eta = r - 2s,\]

\[(E2')\quad \eta \cdot \eta = r - 1,\]

and \((E1'), (E2'), (E4),\) and the "reverse Cauchy-Schwarz inequality" imply

\[
0 \leq (r - 2s)^2 - (\xi \cdot \xi)(r - 1) = \Delta \\
\leq (r - 2s)^2 - (r + 1)(r - 1) = 1 - 4s(r - s) \leq 1,
\]

hence \((E3'')\).

By changing the sign of \(\eta\) if necessary, we assume that \(s = 0\).

From the above evaluation of \(\Delta\), we also obtain either of the following:

\[(F1)\quad \Delta = r^2 - (\xi \cdot \xi)r + (\xi \cdot \xi) = 0,\]

\[(F2)\quad \Delta - 1 = r^2 - (\xi \cdot \xi)r + (\xi \cdot \xi - 1) = 0.\]

Case \((F1)\) is impossible to occur, since its discriminant, \((\xi \cdot \xi)^2 - 4(\xi \cdot \xi)\), cannot be the square of any integer because of hypothesis (i). Thus we have

\[(F2')\quad r = 1 \quad \text{or} \quad r = \xi \cdot \xi - 1.\]

However, \((F2')\) contradicts hypothesis (ii), since the simultaneous equations

\[\xi \cdot \eta = \xi \cdot \xi - 1, \quad \eta \cdot \eta = \xi \cdot \xi - 2\]

are equivalent to

\[\xi \cdot (\xi - \eta) = 1, \quad (\xi - \eta) \cdot (\xi - \eta) = 0.\]

This contradiction completes the proof. \(\square\)

2.2. Facts. Let \(M\) be \(\mathbb{CP}^2 \# n \mathbb{CP}^2\) with \(n \geq 2\), and \(O(M)\) the orthogonal group of the intersection form of \(M\). Recall the following facts: cf. [W2, 1.5, 1.6; W3, Lemma 2 and Corollary to Theorem 2].
(1) Orientation-preserving diffeomorphisms of $M$ realize the following reflections in $O(M)$: for $\xi = (b_0; b_1, \ldots, b_n) \in H_2(M; \mathbb{Z})$,

$$R_i: \xi \rightarrow (b_0; b_1, \ldots, -b_i, \ldots, b_n), \quad 0 \leq i \leq n.$$ $$R_{jk}: \xi \rightarrow (b_0; b_1, \ldots, b_k, \ldots, b_j, \ldots, b_n), \quad 1 \leq j < k \leq n.$$ $$R: \xi \rightarrow \xi + \begin{cases} 2(b_0 - b_1 - b_2)(1; 1, 1), & n = 2, \\ (b_0 - b_1 - b_2 - b_3)(1; 1, 1, 1, 0, \ldots, 0), & n \geq 3. \end{cases}$$

(2) Reflections $R_i$, $R_{jk}$, and $R$ above generate $O(M)$ if $2 \leq n \leq 9$.

**Criterion 2.3.** Let $\xi$ be a class in $H_2(CP^2#n\overline{CP^2}; \mathbb{Z})$, $n \geq 2$.

(1) If $\xi \cdot \xi > 0$, then $\xi$ is transformed, with reflections $R_i$, $R_{jk}$, and $R$ of 2.2(1), into $(a; b_1, \ldots, b_n)$ such that

$$a > b_1 \geq \cdots \geq b_n \geq 0,$$ $$a \geq \begin{cases} b_1 + b_2, & n = 2, \\ b_1 + b_2 + b_3, & n \geq 3. \end{cases}$$

(2) If $\xi \cdot \xi \geq 5$ and $(a; b_1, \ldots, b_n)$ in (1) satisfies

$$(a - 1)^2 > b_1^2 + \cdots + b_n^2,$$

then $\xi$ is not representable by $S^2$.

**Proof.** (1) The case for $n = 2$ is nothing but 2.3 of [W2]. Set $n \geq 3$ and $\xi = (a; b_1, \ldots, b_n)$. Letting $R_i$ and $R_{jk}$ act if necessary, assume that

$$a > b_1 \geq \cdots \geq b_n \geq 0.$$ 

Then observe that $R\xi = (2a - b_1 - b_2 - b_3; \ldots)$ and that

$$a < b_1 + b_2 + b_3 \quad \text{implies} \quad |2a - b_1 - b_2 - b_3| < a,$$

thus obtaining the assertion.

(2) By virtue of Lemma 2.1, it is enough to show that there exists no class $\eta \in H_2(CP^2#n\overline{CP^2}; \mathbb{Z})$ such that $\xi \cdot \eta = 1$ and $\eta \cdot \eta = 0$. Suppose that $\eta$ is such a class. Without loss of generality, assume that

$$\xi = (a; b_1, \ldots, b_n) = (a; b), \quad |a| - |b| > 1, \quad b \in \mathbb{Z}^n.$$ 

Note that for $\eta = (a'; b') \in \mathbb{Z} \times \mathbb{Z}^n$, $|a'| = |b'| \geq 1$. Then, the “ordinary” Cauchy-Schwarz inequality for $\mathbb{Z}^n$ gives

$$1 = |\xi \cdot \eta| \geq (|a| - |b|)|a'| > 1,$$

a contradiction. □

**2.4. Proof of Theorem 1.1.** Taking a connected sum with $\overline{CP^2}$ if $n = 2$, assume that $n = 3$. As in 2.3(1), transform a given class $\xi$, with $R_i$, $R_{jk}$, and $R$, into $(a; b, c, d)$ such that

$$a > b \geq c \geq d \geq 0, \quad a \geq b + c + d.$$ 

Then, it is easy to verify the following:

(1) $a - b \geq 2$ implies $(a - 1)^2 > b^2 + c^2 + d^2$;

(2) $a - b \geq 2$ and $\xi \cdot \xi \leq 4$ imply $(a; b, c, d) = (2; 0, 0, 0)$. 

Hence, Theorem 1.1 follows immediately from (1), (2), 2.3(2), 2.2, and the fact that the classes \((k + 1; k, 0, 0), (k + 1; k, 1, 0),\) and \((2; 0, 0, 0)\) are representable by \(S^2\); cf. [W3, §1; L1, §2]. □

2.5. Remarks. (1) The problem of representing homology classes by embedded 2-spheres for a closed connected oriented 4-manifold \(M\) with \(b^+ = 1\) was completely solved only for \(M = \mathbb{CP}^2[T], S^2 \times S^2[K],\) and \(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\) [L1, Lu]. Note that those results are now corollaries to Theorem 1.1: reverse the orientation of \(M\) if necessary, blow up a point or two of \(M\), and apply Theorem 1.1.

(2) Let \(M\) be either \(S^2 \times S^2\) or \(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\). Recall [W3, Lemma 1] that the latter can be considered as the twisted 2-sphere bundle over \(S^2, S^2 \times S^2\), while the former is the other trivial one. It is interesting to observe from this point of view that \(\xi \in H_2(M; \mathbb{Z})\) is representable by \(S^2\) if and only if \(\xi\) corresponds to one of the following three cases: fiber(s), fiber(s) with one cross section, two cross sections for \(M = S^2 \times S^2\).

(3) Lawson treated only the following classes in \(H_2(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}; \mathbb{Z})\):
\[
(2k + 3; 2k + 1, 2m), \quad k \geq m \geq 1, \quad \text{square} \geq 5 \quad [L1, \text{Theorem 4}];
\]
\[
(2k + 2; 2k, 2m + 1), \quad k \geq m \geq 1, \quad \text{square} \geq 11 \quad [L2, \text{Proposition 5}].
\]
Note that such a class is reduced with our procedure 2.3(1) to one of
\[
(2k' + 3; 2k' + 1, 0), \quad (2k' + 3; 2k' + 1, 2), \quad (2k' + 2; 2k', 1);
\]
and therefore is forbidden by Theorem 1.1 to be represented by an embedded 2-sphere.

(4) From a result of Wall [W3, p. 138] and a result of Hirai [H, Theorem 7], it follows that if \(\xi \in H_2(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}; \mathbb{Z})\) is ordinary with \(-12 < \xi \cdot \xi < 8\), then \(\xi\) is representable by \(S^2\). According to Li [Li, Theorem 6], it turns out that if \(\xi \in H_2(\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}; \mathbb{Z}), 2 \leq n \leq 9\), is ordinary with \(\xi \cdot \xi = 0\), then \(\xi\) is representable by \(S^2\). On the other hand, note that for any integer \(r\), there is a class \(\xi \in H_2(\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}; \mathbb{Z}), n \geq 2,\) representable by \(S^2\) with \(\xi \cdot \xi = r\): e.g., \(\xi = (k + 1; k, 0, 0, \ldots, 0)\) or \((k + 1; k, 1, 0, \ldots, 0)\).

3. Characteristic 2-spheres in 4-manifolds

This section is devoted to the proofs of Theorems 1.2 and 1.3. Let us begin by introducing an extended version of the 11/8 conjecture (cf. [Mt, §1; FM, §4]).

**Conjecture 3.1.** The signature of a closed oriented spin 4-manifold \(M\) must satisfy the inequality \(b_2(M) \geq (11/8)|\sigma(M)|\).

**3.2. Notation.** For an integer \(r > 0\), let \(\text{CONJ}(r, P)\) denote the proposition that Conjecture 3.1 is true of any closed connected oriented spin 4-manifold \(M\) with \(b^+ < r\) and with a particular property \(P\) on \(H_1(M; \mathbb{Z})\).

**3.3. Remarks.** (1) By Rohlin's signature theorem [R], the inequality \(b_2 \geq (11/8)|\sigma|\) in Conjecture 3.1 is equivalent to
\[
b^+ = b^+ + 16m, \quad m \in \mathbb{Z}, \quad -[b^+/19] \leq m \leq [b^+/3].
\]

(2) Donaldson's Theorem 2 [D] is equivalent to \(\text{CONJ}(3, (\ast))\), where \(\ast\) is the property that \(H_1(M; \mathbb{Z})\) has no 2-torsion.
Lemma 3.4. Let $M$ be a closed connected oriented 4-manifold with $b^+ \leq r$ and with a particular property $P$ on $H_1(M; \mathbb{Z})$, and $\xi \in H_2(M; \mathbb{Z})$ a characteristic homology class representable by $S^2$. Then $\text{CONJ}(r, P)$ implies $\xi \cdot \xi = \sigma(M) + 16m$ for some integer $m$ such that

1. if $b^+ = r$, then $m \leq \max\{[(r-1)/3], [(b^- - r)/16]\}$;
2. if $b^+ < r$, then $m \leq \max\{[b^+/3], [(b^- - b^+ - 2)/16]\}$.

Proof. Note that Kervaire and Milnor's generalization [KM, Theorem 1] of Rohlin's signature theorem implies $\xi \cdot \xi = \sigma(M) + 16m$ for some integer $m$.

1. Suppose that $m$ does not satisfy the above inequality: namely,

   $m > \max\{[(r-1)/3], [(b^- - r)/16]\}$.

   Then $\xi \cdot \xi = r - b^- + 16m > 0$. Let a 2-sphere $S \subset M$ represent $\xi$, "blow up" $\xi \cdot \xi - 1$ $(\geq 0)$ distinct point(s) of $S$, and "blow down" the resulting "exceptional curve" of self-intersection $+1$: namely,

   $$(M, S)\#(\xi \cdot \xi - 1)\mathbb{CP}^2, \mathbb{CP}^1) \cong (N, \phi)\#(\mathbb{CP}^2, \mathbb{CP}^1),$$

   thereby constructing a closed connected oriented spin 4-manifold $N$ with the property $P$ on $H_1(N; \mathbb{Z})$ and with $b^+(N) = r - 1$, $b^-(N) = b^+(N) + 16m$, and $m > [b^+(N)/3]$. By 3.3(1), the existence of such an $N$ is contradictory to $\text{CONJ}(r, P)$.

2. The proof is similar. Suppose not. Let a 2-sphere $S \subset M$ represent $\xi$, "blow up" $\xi \cdot \xi + 1$ $(\geq 0)$ distinct point(s) of $S$, and "blow down" the resulting "exceptional curve" of self-intersection $-1$, thereby producing a spin 4-manifold whose existence is contradictory to $\text{CONJ}(r, P)$. □

3.5. Proof of Theorem 1.3. Let $S \subset M$ be a 2-sphere representing $\xi$.

1. immediately follows from 3.3(2) and Lemma 3.4(1).
2. By taking the connected sum $(M, S)\#(2 - b^+)\mathbb{CP}^2, \mathbb{CP}^1)$, reduce the case to $b^+ = 2$, to obtain the assertion through 3.3(2) and Lemma 3.4(2).
3. First consider the case $b^+ = b^- = 3$. Note that (1) implies $\xi \cdot \xi \leq 0$. Reversing the orientation of $M$, observe that $\xi \cdot \xi = 0 = \sigma(M)$. Next suppose either $b^+ < 3$ or $b^- < 3$. Take the connected sum

   $$(M, S)\#(3 - b^+)\mathbb{CP}^2, \mathbb{CP}^1)\#(3 - b^-)(\mathbb{CP}^2, \mathbb{CP}^1),$$

   thereby obtaining $\xi \cdot \xi - b^+ + b^- = 0$, i.e., $\xi \cdot \xi = b^+ - b^- = \sigma(M)$. □

3.6. Proof of Theorem 1.2. As for (2), the "if" (resp. "only if") part is contained in (1) (resp. Theorem 1.3(3)). Thus, in virtue of 2.2, it is enough to show that any characteristic homology class

$$\xi = (a; b_1, \ldots, b_n) \in H_2(\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}; \mathbb{Z}), \quad 3 \leq n \leq 9, \quad \xi \cdot \xi = 1 - n,$$

is transformed, with reflections $R_i, R_{jk}$, and $R$, into $(1; 1, \ldots, 1)$, which is representable by $S^2$. First note that $a$ and $b_i$ are odd, and that

$$a^2 - b_1^2 - \cdots - b_n^2 = 1 - n.$$

Next assume, by letting $R_i$ and $R_{jk}$ act if necessary, that

$$a \geq b_1 \geq \cdots \geq b_n \geq 1.$$
Then, it is easy to verify that if \( a > 1 \), then
\[
a < b_1 + b_2 + b_3 < 3a;
\]
and hence that if \( a > 1 \), then
\[
|2a - b_1 - b_2 - b_3| < a.
\]
Therefore, reflection \( R \), which transforms \( \xi \) into \((2a - b_1 - b_2 - b_3; \ldots)\), strictly reduces \( a \) whenever \( a > 1 \). This completes the proof. \( \Box \)

3.7. Remarks. (1) Because of Theorem 1.3(3), Theorem 1.2(2) is generalized as follows. Let \( M \) be \( mCP^2\# n\overline{CP^2} \) or \( r(S^2 \times S^2), \ 0 \leq m, n, r \leq 3 \): when \( m = n = 0 \) or \( r = 0 \), \( M \) is understood to be \( S^4 \). Suppose that \( \xi \in H_2(M; \mathbb{Z}) \) is characteristic. Then \( \xi \) is representable by \( S^2 \) if and only if \( \xi \cdot \xi = \sigma(M) \). Confer [K, Theorem 1; L1, Theorems 1, 2; Lu, Theorem; H, Theorems 1, 2, 3; Li, §2].

(2) There are 4-manifolds \( M \) that possess characteristic homology classes that are representable by \( S^2 \) but have self-intersection different from the signature of \( M \). For example, let \( M \) be \( 19n(CP^2\# CP^2) \), \( n \geq 1 \). Recall [Mn, Theorem 2.4] that a \( K3 \) surface \( K \) is almost completely decomposable, i.e., that \( K\# CP^2 \) is diffeomorphic to \( 4CP^2\# 19\overline{CP^2} \). Hence, \( M \) is diffeomorphic to \( n(K\# 16CP^2) \). It follows from this fact, together with Theorem 4 of [W1] and Corollary to Theorem 2 of [W3], that if \( \xi \) is a primitive characteristic homology class in \( H_2(M; \mathbb{Z}) \) with \( \xi \cdot \xi = 16i, \ |i| \leq n \), then \( \xi \) is representable by \( S^2 \) (cf. [W4, (4.7)]).

We conclude by stating a corollary to Theorem 1.3, which contains a result of Lawson [L1, Theorem 5] (cf. [Li, §3]).

Corollary 3.8. Let \( M \) be a closed 1-connected oriented 4-manifold.

(1) If \( b \leq 2 \), then the normal Euler number of a characteristic embedding of \( RP^2 \) in \( M \) is either of \( \sigma(M) \pm 2 \), both of which can be realized for \( M = mCP^2\# n\overline{CP^2} \) or \( r(S^2 \times S^2), \ 0 \leq m, n, r \leq 2 \).

(2) If \( b \leq 1 \), then the normal Euler number of a characteristic embedding of Klein’s bottle \( Kl \) in \( M \) is one of \( \sigma(M) \) and \( \sigma(M) \pm 4 \), all of which can be realized for \( M = mCP^2\# n\overline{CP^2} \) or \( r(S^2 \times S^2), \ 0 \leq m, n, r \leq 1 \).

Proof. (1) (resp.(2)) immediately follows from Theorem 1.3(3), 3.7(1) and Rohlin-Lawson’s “Connecting Lemma” of [L1,§5], which asserts that if \( \nu \) is the normal Euler number of a characteristic embedding of \( RP^2 \) (resp. \( Kl \)) in a closed 1-connected oriented 4-manifold \( M \), then there is an embedding of \( S^2 \) into \( M\#(S^2 \times S^2) \) (resp. \( M\#2(S^2 \times S^2) \)), representing a characteristic homology class \( \xi \) with \( \xi \cdot \xi \) equal to either of \( \nu \pm 2 \) (resp. one of \( \nu \) and \( \nu \pm 4 \)). \( \Box \)

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Department of Mathematical Science, University of Tokyo, Hongo, Tokyo 113, Japan