POSITIVE HARMONIC MAJORIZATION OF THE REAL PART OF A HOLOMORPHIC FUNCTION

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Abstract. Let $U$ be the unit disc. This paper investigates which domains $D$ in the complex plane have the property that $\Re f$ belongs to $h^1$, or the more restrictive property that $e^f$ belongs to the Smirnov class $\mathcal{N}^+$, for every holomorphic function $f: U \to D$.

1. Introduction

For each domain (i.e., connected open set) $D$ in $\mathbb{C}$, let $\mathcal{H}(U, D)$ be the class of all holomorphic functions from the open unit disc $U$ into $D$. As usual, let $\mathcal{N}$ be the Nevanlinna class of all holomorphic functions $f$ on $U$ for which

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta < \infty.$$ 

Each function $f$ in $\mathcal{N}$ has a nontangential limit, denoted by $f(e^{i\theta})$, at almost every boundary point $e^{i\theta}$. The Smirnov class $\mathcal{N}^+$ is the subclass of functions $f$ in $\mathcal{N}$ for which

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \to \int_0^{2\pi} \log^+ |f(e^{i\theta})| \, d\theta \quad (r \to 1-).$$

A discussion of the classes $\mathcal{N}$ and $\mathcal{N}^+$ can be found in Garnett [5, Chapter II].

Theorem A below is classical; see Helms [6, Theorem 8.33] for the "if" assertion and Frostman [4, §52] or Nevanlinna [8, VII, §4.2] for the converse. Theorem B was established more recently by Ahern and Cohn [1]. For an introduction to the notion of thin sets the reader is referred to [6, Chapter 10].

Theorem A. Let $D$ be a domain in $\mathbb{C}$. Then $f \in \mathcal{N}$ for every $f$ in $\mathcal{H}(U, D)$ if and only if $\partial D$ has positive logarithmic capacity.

Theorem B. Let $D$ be a domain in $\mathbb{C}$. Then $f \in \mathcal{N}^+$ for every $f$ in $\mathcal{H}(U, D)$ if and only if $\mathbb{C} \setminus D$ is nonthin at $\infty$.

In this paper we investigate which domains $D$ have the property that $e^f \in \mathcal{N}$, or that $e^f \in \mathcal{N}^+$, for every $f$ in $\mathcal{H}(U, D)$. We note that $e^f \in \mathcal{N}$ if

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and only if the real part of \( f \) can be written as the difference of two positive harmonic functions on \( U \), i.e., \( \Re f \in h^1 \). Further (assuming that \( e^f \in \mathcal{N}^+ \)), \( e^f \in \mathcal{N}^+ \) if and only if \( \Re f \) is majorized in \( U \) by the Poisson integral of its nontangential boundary values.

We will denote the right half-plane by \( D_0 \). It is obviously the case that if \( D \subseteq D_0 \) then \( \Re f \in h^1 \) for every \( f \) in \( \mathcal{H}(U, D) \). The following result describes the situation for simply connected domains that contain \( D_0 \).

**Theorem 1.** Let \( D \) be a simply connected domain that contains \( D_0 \). Then \( \Re f \in h^1 \) for every \( f \) in \( \mathcal{H}(U, D) \) if and only if

\[
\int_{-\infty}^{\infty} \frac{\text{dist}(iy, \partial D)}{1 + y^2} \, dy < \infty.
\]

As will be seen in §2, Theorem 1 follows easily from a known result on the angular derivative problem. Now suppose that \( D \) is a simply connected domain that contains \( D_0 \) and satisfies (1). If \( D_1 \) is a domain (not necessarily simply connected) contained in \( D \), then clearly \( \Re f \in h^1 \) (or, equivalently, \( e^f \in \mathcal{N}^+ \)) for every \( f \) in \( \mathcal{H}(U, D_1) \). The following result identifies which of these domains have the stronger property that \( e^f \in \mathcal{N}^+ \) for every \( f \in \mathcal{H}(U, D_1) \).

**Theorem 2.** Let \( D \) be a simply connected domain that contains \( D_0 \) and satisfies (1), and let \( D_1 \) be a domain contained in \( D \). Then \( e^f \in \mathcal{N}^+ \) for every \( f \) in \( \mathcal{H}(U, D_1) \) if and only if \( \mathbb{R}^4 \setminus D_1^* \) is nonthin at \( \infty \), where

\[
D_1^* = \{(x_1, \ldots, x_4) \in \mathbb{R}^4 : (x_1^2 + x_2^2 + x_3^2)^{1/2} + ix_4 \in D_1\}.
\]

Here \( \infty \) denotes the Alexandroff point for \( \mathbb{R}^4 \). The condition "\( \mathbb{R}^4 \setminus D_1^* \) is nonthin at \( \infty \)" is equivalent to "\( D_0 \setminus D_1 \) is not minimally thin at \( \infty \) with respect to \( D_0 \)", but the proof of Theorem 2 (see §3) does not use any results concerning minimally thin sets.

Let \( PI[g] \) denote the Poisson integral in \( U \) of a function \( g \) in \( L^1(\partial U) \). The following is a simple consequence of Theorem 2.

**Corollary.** Let \( D_1 \) be a domain contained in \( D_0 \). Then \( u = PI[u|_{\partial U}] \) for every \( f = u + iv \) in \( \mathcal{H}(U, D_1) \) if and only if \( \mathbb{R}^4 \setminus D_1^* \) is nonthin at \( \infty \).

2. Proof of Theorem 1

2.1. The following result is due to Oikawa [9], who formulated it in terms of an infinite strip rather than a half-plane. A closely related result had previously been given by Rodin and Warschawski [10, Theorem 2].

**Theorem C.** Let \( D \) be a simply connected domain that contains \( D_0 \). Then (1) holds if and only if there is a one-to-one conformal map \( g \) of \( D \) onto \( D_0 \) such that \( g(z)/z \) has a finite nonzero limit as \( |z| \to \infty \) in \( \{re^{i\theta} : |\theta| < \theta_0 \} \) for each \( \theta_0 \) in \( (0, \pi/2) \).

The "if" part of Theorem 1 follows easily. To see this, let \( D \) be as in the statement of Theorem 1 and suppose (1) holds. Then there is a function \( g \) as in Theorem C. Let \( z = x + iy \), and put \( l = \lim_{x \to \infty} g(x)/x \). Clearly \( l \) is real, so \( l \in (0, \infty) \). Thus \( \Re g \) is a positive harmonic function on \( D \) whose Poisson integral representation in \( D_0 \) includes the term \( lx \). Hence \( \Re g \) majorizes \( lx \) on \( D_0 \). It follows that if \( f \in \mathcal{H}(U, D) \) the function \( l^{-1} \Re g \circ f \) is a positive harmonic majorant of \( \Re f \), and this implies that \( \Re f \in h^1 \).
2.2. Conversely, suppose that \( D \) is a simply connected domain that contains \( D_0 \) and that \( \Re f \in h^1 \) for every \( f \) in \( \mathcal{H}(U, D) \). It is certainly not the case that \( \Re f \in h^1 \) for every holomorphic function on \( U \) (see below), so \( D \neq C \). Thus we can choose \( f \) to be a one-to-one conformal mapping of \( U \) onto \( D \). Let \( f = u + iv \). By hypothesis, \( u \) has a positive harmonic majorant, \( h \) say, on \( U \). Since \( h \circ f^{-1}(z) \geq u \circ f^{-1}(z) = x \), the positive harmonic function \( H = h \circ f^{-1} \) majorizes \( x \) on \( D \). We define \( \phi: \mathbb{R} \to [0, \infty) \) by \( \phi(y) = \text{dist}(iy, \partial D) \). If \( \phi(y) > 0 \) then \( D \) contains the open disc of centre \( iy \) and radius \( \phi(y) \). Thus, applying Harnack’s inequalities, we obtain
\[
H(iy) \geq C \phi(y)/2 + iy \geq C\phi(y)/2,
\]
where \( C \) is a positive constant. Since
\[
\int_{\{\phi(y) > 0\}} H(iy)/(1 + y^2) \, dy < \infty,
\]
it is now clear that (1) holds.

3. Proof of Theorem 2

3.1. We recall some definitions. A positive harmonic function is called quasi-bounded if it can be expressed as the limit of an increasing sequence of bounded positive harmonic functions. Let \( W \) be an open subset of \( \mathbb{R}^n \) \((n \geq 2)\), let \( s \) be a positive superharmonic function on \( W \), and let \( A \subseteq W \). Then the reduced function (or réduite) of \( s \) relative to \( A \) in \( W \) is defined to be the infimum of all positive superharmonic functions \( S \) on \( W \) that satisfy \( S \geq s \) on \( A \). A subset \( A \) of \( \mathbb{R}^n \) \((n \geq 3)\) is said to be thin at \( \infty \) if the reduced function of (the constant function) \( 1 \) relative to \( A \) in \( \mathbb{R}^n \) is less than \( 1 \) at some point of \( \mathbb{R}^n \).

The following lemma is an immediate consequence of Huber [7, Lemma].

**Lemma A.** Let \( A \subseteq D_0 \). The following are equivalent:

(i) the reduced function of \( z \to x \) relative to \( A \) in \( D_0 \) equals \( x \);

(ii) the set \( A^* = \{ (x_1, \ldots, x_n) : (x_1^2 + x_2^2 + x_3^2)^{1/2} + ix_4 \in A \} \) is nonthin at \( \infty \).

3.2. Now let \( D \) be as in the statement of Theorem 2. It follows (see §2.1) that the subharmonic function \( x^+ \) has a harmonic majorant in \( D \). Let \( h \) denote the least harmonic majorant of \( x^+ \) in \( D \). Suppose that \( \mathbb{R}^4 \setminus D_1^* \) (and hence also \( (D_0 \setminus D_1)^* \)) is nonthin at \( \infty \). It follows from Lemma A that if \( s \) is any positive superharmonic function on \( D \) that majorizes \( h \) on \( D \setminus D_1 \) so that \( s(z) \geq x \) on \( D_0 \setminus D_1 \), then \( s(z) \geq x \) on \( D_0 \). Hence \( s \) is a superharmonic majorant of \( x^+ \) on \( D \), and so \( s \geq h \) on \( D \). Thus the reduced function of \( h \) relative to the set \( D \setminus D_1 \) in \( D \) equals \( h \) itself, and so (see Doob [3, 1.VIII.10])

\[
h(z) = \int_{D \cap \partial D_1} h(w) \, d\mu_{z, D_1}(w) = \lim_{m \to \infty} h_m(z) \quad (z \in D_1)
\]

where

\[
h_m(z) = \int_{D \cap \partial D_1} \min\{h(w), m\} \, d\mu_{z, D_1}(w) \quad (z \in D_1)
\]

and \( d\mu_{z, D_1} \) denotes harmonic measure for \( D_1 \) and \( z \). Since \( h \) is a positive harmonic function on \( D_1 \) that majorizes \( x \) there, it follows that if \( f \in \mathcal{H}(U, D_1) \) then \( h \circ f \) is a positive harmonic function on \( U \) that majorizes the real part of \( f \). Further, if we define \( u_m = h_m \circ f + \Re f - h \circ f \), then each harmonic function \( u_m \) is bounded above and so is majorized on \( U \) by the Poisson integral of its (nontangential) boundary values. It follows, on letting
\( m \to \infty \), that \( \Re f \) is majorized in \( U \) by the Poisson integral of its boundary values. Thus \( e^f \in \mathcal{N}^+ \) for every \( f \) in \( \mathcal{H}(U, D_1) \).

3.3. To prove the converse, let \( f = u + iv \), where \( f : U \to D_1 \) is the covering map (see Ahlfors [2, Chapter 10]). If \( D_1 \) is bounded, the result is trivial. If \( D_1 \) is unbounded then

\[
\int_0^{2\pi} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \, g(f(e^{i\theta})) \, d\theta = \int_{\partial D_1} g \, d\mu_{f(w), D_1} \quad (w \in U)
\]

for any continuous function \( g \) on \( D_1 \cup \{\infty\} \). If we put \( g(z) = \min\{x^+, m/|z|\} \) in (2) and let \( m \) tend to infinity, it follows that

\[
u^+(w) \leq \int_0^{2\pi} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \, u^+(e^{i\theta}) \, d\theta = \int_{\partial D_1} x^+ \, d\mu_{f(w), D_1}(z) \quad (w \in U)
\]

in view of the hypothesis that \( e^f \in \mathcal{N}^+ \). Since \( f(U) = D_1 \), we have

\[(\Re w)^+ \leq \int_{\partial D_1} x^+ \, d\mu_{w, D_1}(z) \quad (w \in D_1).\]

It follows that \( x^+ \) has a quasi-bounded harmonic majorant on \( D_1 \) and hence on \( D_1 \cap D_0 \). Thus \( x \) is a quasi-bounded harmonic function on \( D_1 \cap D_0 \), and so

\[\Re w = \int_{D_0 \cap \partial D_1} x \, d\mu_{w, D_1 \cap D_0}(z) \quad (w \in D_1 \cap D_0).\]

Hence the reduced function of \( x \) relative to \( D_0 \setminus D_1 \) in \( D_0 \) equals \( x \) itself. It follows from Lemma A that \( \mathbb{R}^4 \setminus D_1^+ \) is nonthin at \( \infty \), and this completes the proof of Theorem 2.

References