DISTORTION OF SETS BY INNER FUNCTIONS

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Abstract. For any inner function $f$ with $f(0) = 0$ and any Borel set $E \subset \mathbb{D}$

\[ M_\alpha(z \in \mathbb{D} : f(z) \in E) \geq M_\alpha(E), \quad 0 < \alpha \leq 1, \]

where $M_\alpha$ denotes $\alpha$-dimensional Hausdorff measure. In the case that $0 < M_\alpha(E) < \infty$ we have equality only for rotations of the identity.

1. Introduction

Let $f(z)$ be an inner function, i.e., a bounded analytic function on the unit disk $\mathbb{D}$ with radial boundary values of modulus 1 (a.e.). Piranian and Weitsman [5] asked about the length of the level sets of $f$. Belna et al. [1] gave an example of a Blaschke product $f$ and a line $L$ so that $\{ z : |z| < 1, f(z) \in L \}$ has infinite length. Jones [4] gave an example of an inner function $f$ so that $\{ z : |z| < 1, f(z) \in L \}$ has infinite length for a continuum of lines $L$. We give a lower bound for functions with normalisation $f(0) = 0$, obtaining

\[ \text{Length}(f^{-1}(L)) \geq \text{Length}(L). \]

In fact we prove a general inequality for arbitrary sets involving the $\alpha$-dimensional Hausdorff measure $M_\alpha$.

Theorem. For any inner function $f$ with $f(0) = 0$ and any Borel set $E \subset \mathbb{D}$

\[ M_\alpha(z \in \mathbb{D} : f(z) \in E) \geq M_\alpha(E), \quad 0 < \alpha \leq 1. \]

In the case that $0 < M_\alpha(E) < \infty$ we have equality only for rotations of the identity.

Remarks. 1. Simple examples show that the theorem fails for $\alpha > 1$.

2. This is reminiscent of a result of Fernandez and Pestana [2] who prove

\[ \text{Length}(f^{-1}(L)) \geq C_\alpha \text{Length}(L), \]

where $C_\alpha$ is a positive constant depending on $\alpha$. The case $\alpha = 1$

\[ M_1(z \in \partial \mathbb{D} : \exists \lim_{r \to 1} f(rz) \in E) = M_1(E) \]

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is due to Lowner; see Tsuji [6]. Other similar results for univalent functions have been given by the author [3]. We shall not consider these radial boundary results here; however, we conjecture that the results of Fernandez and Pestana can be given with $C_\alpha = 1$.

2. AN INEQUALITY

Everything is based on

**Lemma 1.** Let $f(z)$ be an inner function with $f(0) = 0$. For all $w$, $|w| < 1$, except for possibly a closed set of zero logarithmic capacity,

$$\sum_{n=1}^{\infty} \frac{1}{|f'(a_n)|} \geq 1,$$

where $a_n$ are the roots of $f(z) = w$.

We rely on Frostman's Theorem:

**Lemma 2.** Let $f(z)$ be an inner function with $f(0) = 0$. For all $w$, $|w| < 1$, except for possibly a set of zero logarithmic capacity,

$$\frac{f(z) - w}{1 - \overline{w} f(z)} = c_0 \prod_{n=1}^{\infty} \frac{-\overline{a_n}}{|a_n|} \left\{ \frac{z - a_n}{1 - \overline{a_n} z} \right\}$$

where $a_n$ are the roots of $f(z) = w$, counting multiple values, and where $c_0$ is a constant of modulus 1.

Thus except for a set of zero capacity

$$f(z) = \left( c_0 \prod_{n=1}^{\infty} \frac{-\overline{a_n}}{|a_n|} \left\{ \frac{z - a_n}{1 - \overline{a_n} z} \right\} + w \right) \left/ \left( 1 + \overline{w} c_0 \prod_{n=1}^{\infty} \frac{-\overline{a_n}}{|a_n|} \left\{ \frac{z - a_n}{1 - \overline{a_n} z} \right\} \right) \right.$$

But as $f(0) = 0$ we get $|w| = \prod_{n=1}^{\infty} |a_n|$.

Next we make use of

**Lemma 3.** For any $Y$, $0 < Y < 1$, and sequence $X_n$, $0 < X_n < 1$, with $Y = \prod_{n=1}^{\infty} X_n$, we have $\sum_{n=1}^{\infty} (1 - X_n^2) \geq 1 - Y^2$ and equality occurs only if all the $X_n$ except at most one are 1.

The proof is left to the reader.

We continue with the proof of Lemma 1. A well-known consequence of the Schwarz Lemma is that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

for any analytic function $f$ mapping the unit disk into itself. Thus summing over all $a_n$ such that $f(a_n) = w$ gives

$$\sum_{n=1}^{\infty} \frac{1}{|f'(a_n)|} \geq \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{1 - |w|^2} \geq 1,$$

by Lemma 3 and inequality (2). This proves (1) except maybe for a set, not necessarily closed, of capacity zero. Now in (2) equality is only possible if
$f(z) = cz$, $|c| = 1$, and then the exceptional set is empty. Hence for any other inner function $f$ and some $w_0$ satisfying (1) we get

$$\sum_{n=1}^{N} \frac{1}{|f'(a_n)|} > 1,$$

for some $N$ depending on $w_0$ (and $f$). Now on the Riemann surface of $f$, $a_n = a_n(w)$ is analytic in $w$. Consequently there is a neighbourhood of $w_0$ where (3) holds. (True even if one of the $a_n$ give $f'(a_n) = 0$.) This completes the proof of Lemma 1.

3. Proof of the Theorem

It is easy to see that if $f$ is Lipschitz and 1:1 on a metric space $A$ then $M_\alpha(f(A)) \leq (\text{Lip}(f))^\alpha M_\alpha(A)$ for all $\alpha > 0$. Hence for a nonconstant analytic function $f$ on $D$ we may partition $D$ into a countable number of "open-closed" triangles $T$ upon which $|f'|$ is very nearly constant and equal to $\text{Lip}_T(f)$. Hence summing and taking the limit we have

**Lemma 4.** With the previous notation for any $\alpha > 0$, and any Borel set $E$

$$M_\alpha(z \in D : f(z) \in E) \geq \int_E \left\{ \sum_{f(a_n) = w} \frac{1}{|f'(a_n)|^\alpha} \right\} dM_\alpha(w).$$

Now for $0 < \alpha \leq 1$ by Lemma 1 we have that the integrand is bounded below by 1 except for a set of zero logarithmic capacity and hence $\alpha$-dimensional Hausdorff measure zero. For if only one term of the sum is more than 1 then we are done, otherwise we have all the terms bounded above by 1 in which case as $0 < \alpha \leq 1$

$$\sum_{f(a_n) = w} \frac{1}{|f'(a_n)|^\alpha} \geq \sum_{f(a_n) = w} \frac{1}{|f'(a_n)|} \geq 1.$$

Hence we have the inequality of the theorem. To obtain equality we need equality in Lemma 1 (and Lemma 3) over a set $W$ of finite $\alpha$-Hausdorff measure. It follows from Lemma 2 that for $w \in W$, $f(z) = w$ has exactly one root $z$, $|z| = |w|$. Thus $f(z) = cz$ for some $c$, $|c| = 1$. This completes the proof of the theorem.

**References**


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