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DISTORTION OF SETS BY INNER FUNCTIONS

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ABSTRACT. For any inner function f with $f(0) = 0$ and any Borel set $E \subset \mathbf{D}$

$$M_\alpha(z \in \mathbf{D} : f(z) \in E) \geq M_\alpha(E), \qquad 0 < \alpha \leq 1,$$

where M_α denotes α-dimensional Hausdorff measure. In the case that $0 < M_\alpha(E) < \infty$ we have equality only for rotations of the identity.

1. INTRODUCTION

Let $f(z)$ be an inner function, i.e., a bounded analytic function on the unit disk $\mathbf{D}$ with radial boundary values of modulus 1 (a.e.). Piranian and Weitsman [5] asked about the length of the level sets of f. Belna et al. [1] gave an example of a Blaschke product f and a line L so that $\{z : |z| < 1, f(z) \in L\}$ has infinite length. Jones [4] gave an example of an inner function f so that $\{z : |z| < 1, f(z) \in L\}$ has infinite length for a continuum of lines L. We give a lower bound for functions with normalisation $f(0) = 0$, obtaining $\text{Length}(f^{-1}(L)) \geq \text{Length}(L)$.

In fact we prove a general inequality for arbitrary sets involving the α-dimensional Hausdorff measure M_α.

Theorem. *For any inner function f with $f(0) = 0$ and any Borel set $E \subset \mathbf{D}$*

$$M_\alpha(z \in \mathbf{D} : f(z) \in E) \geq M_\alpha(E), \qquad 0 < \alpha \leq 1.$$

In the case that $0 < M_\alpha(E) < \infty$ we have equality only for rotations of the identity.

Remarks. 1. Simple examples show that the theorem fails for $\alpha > 1$.

2. This is reminiscent of a result of Fernandez and Pestana [2] who prove that for any Borel set $E \subset \partial\mathbf{D}$ we have

$$M_\alpha\left(z \in \partial\mathbf{D} : \exists \lim_{r \to 1} f(rz) \in E\right) \geq C_\alpha M_\alpha(E), \qquad 0 < \alpha \leq 1,$$

where C_α is a positive constant depending of α. The case $\alpha = 1$

$$M_1\left(z \in \partial\mathbf{D} : \exists \lim_{r \to 1} f(rz) \in E\right) = M_1(E)$$

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is due to Lowner; see Tsuji [6]. Other similar results for univalent functions have been given by the author [3]. We shall not consider these radial boundary results here; however, we conjecture that the results of Fernandez and Pestana can be given with $C_\alpha = 1$.

2. An inequality

Everything is based on

Lemma 1. *Let $f(z)$ be an inner function with $f(0) = 0$. For all w, $|w| < 1$, except for possibly a closed set of zero logarithmic capacity,*

$$\sum_{n=1}^{\infty} \frac{1}{|f'(a_n)|} \geq 1, \tag{1}$$

where a_n are the roots of $f(z) = w$.

We rely on Frostman's Theorem:

Lemma 2. *Let $f(z)$ be an inner function with $f(0) = 0$. For all w, $|w| < 1$, except for possibly a set of zero logarithmic capacity,*

$$\frac{f(z) - w}{1 - \overline{w} f(z)} = c_0 \prod_{n=1}^{\infty} \frac{-\overline{a_n}}{|a_n|} \left\{ \frac{z - a_n}{1 - \overline{a_n} z} \right\}$$

where a_n are the roots of $f(z) = w$, counting multiple values, and where c_0 is a constant of modulus 1.

Thus except for a set of zero capacity

$$f(z) = \left(c_0 \prod_{n=1}^{\infty} \frac{-\overline{a_n}}{|a_n|} \left\{ \frac{z - a_n}{1 - \overline{a_n} z} \right\} + w \right) \Bigg/ \left(1 + \overline{w} c_0 \prod_{n=1}^{\infty} \frac{-\overline{a_n}}{|a_n|} \left\{ \frac{z - a_n}{1 - \overline{a_n} z} \right\} \right).$$

But as $f(0) = 0$ we get $|w| = \prod_{n=1}^{\infty} |a_n|$.
Next we make use of

Lemma 3. *For any Y, $0 < Y < 1$, and sequence X_n, $0 < X_n < 1$, with $Y = \prod_{n=1}^{\infty} X_n$, we have $\sum_{n=1}^{\infty}(1 - X_n^2) \geq 1 - Y^2$ and equality occurs only if all the X_n except at most one are* 1.

The proof is left to the reader.

We continue with the proof of Lemma 1. A well-known consequence of the Schwarz Lemma is that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \tag{2}$$

for any analytic function f mapping the unit disk into itself. Thus summing over all a_n such that $f(a_n) = w$ gives

$$\sum_{n=1}^{\infty} \frac{1}{|f'(a_n)|} \geq \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{1 - |w|^2} \geq 1,$$

by Lemma 3 and inequality (2). This proves (1) except maybe for a set, not necessarily closed, of capacity zero. Now in (2) equality is only possible if

$f(z) = cz$, $|c| = 1$, and then the exceptional set is empty. Hence for any other inner function f and some w_0 satisfying (1) we get

$$\sum_{n=1}^{N} \frac{1}{|f'(a_n)|} > 1, \tag{3}$$

for some N depending on w_0 (and f). Now on the Riemann surface of f, $a_n = a_n(w)$ is analytic in w. Consequently there is a neighbourhood of w_0 where (3) holds. (True even if one of the a_n give $f'(a_n) = 0$.) This completes the proof of Lemma 1.

3. Proof of the Theorem

It is easy to see that if f is Lipschitz and 1:1 on a metric space A then $M_\alpha(f(A)) \leq (\mathrm{Lip}(f))^\alpha M_\alpha(A)$ for all $\alpha > 0$. Hence for a nonconstant analytic function f on $\mathbf{D}$ we may partition $\mathbf{D}$ into a countable number of "open-closed" triangles T upon which $|f'|$ is very nearly constant and equal to $\mathrm{Lip}_T(f)$. Hence summing and taking the limit we have

Lemma 4. *With the previous notation for any $\alpha > 0$, and any Borel set E*

$$M_\alpha(z \in \mathbf{D} : f(z) \in E) \geq \int_E \left\{ \sum_{f(a_n)=w} \frac{1}{|f'(a_n)|^\alpha} \right\} dM_\alpha(w).$$

Now for $0 < \alpha \leq 1$ by Lemma 1 we have that the integrand is bounded below by 1 except for a set of zero logarithmic capacity and hence α-dimensional Hausdorff measure zero. For if only one term of the sum is more that 1 then we are done, otherwise we have all the terms bounded above by 1 in which case as $0 < \alpha \leq 1$

$$\sum_{f(a_n)=w} \frac{1}{|f'(a_n)|^\alpha} \geq \sum_{f(a_n)=w} \frac{1}{|f'(a_n)|} \geq 1.$$

Hence we have the inequality of the theorem. To obtain equality we need equality in Lemma 1 (and Lemma 3) over a set W of finite α-Hausdorff measure. It follows from Lemma 2 that for $w \in W$, $f(z) = w$ has exactly one root z, $|z| = |w|$. Thus $f(z) = cz$ for some c, $|c| = 1$. This completes the proof of the theorem.

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