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THE TOTAL SPLIT CURVATURES OF KNOTTED SPACE-LIKE 2-SPHERES IN MINKOWSKI 4-SPACE

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ABSTRACT. In this paper we establish an analogue of the Fary-Milnor Theorem for space-like 2-spheres in Minkowski 4-space.

INTRODUCTION

A smoothly embedded 2-sphere in $\mathbf{R}^4$ is *unknotted* if it is the boundary of a C^0 embedded 3-disk in $\mathbf{R}^4$. A smoothly embedded 2-sphere in 4-dimensional Minkowski space is *space-like* if the induced metric is positive definite. Such a surface has two S^2-valued Gauss maps. The notion of *split curvature* measures the expected number of preimage of each Gauss map (see [K1]. In this paper we show that if a smoothly embedded space-like 2-sphere is knotted then the sum of its split total curvatures is bounded below by 16π. This can be taken as a Minkowski analogue of the classical Fary-Milnor Theorem for one-dimensional knots in Euclidean 3-space (see [M]).

PRELIMINARIES

Throughout this paper $\mathbf{M}^4$ will denote Minkowski space, the real 4-dimensional vector space equipped with bilinear form $\langle\ ,\ \rangle$ of type (3, 1); that is, the normal form has three plus signs and one minus sign. We shall assume that $\mathbf{M}^4$ is oriented and time oriented; that is, a 4-volume form dV, and future time-like vector field FUT have been chosen. We shall write $\mathrm{LC} = \{v \in \mathbf{M}^4 | \langle v, v\rangle = 0\}$ to denote the light cone in Minkowski space.

A *space-like* 2-*sphere* in $\mathbf{M}^4$ is a smooth embedding $k: S^2 \to \mathbf{M}^4$ such that the induced metric $k^*\langle\ ,\ \rangle$ is positive definite. It follows that the normal bundle is a trivial rank-two bundle with fibres carrying a type $(1, 1)$ metric. To describe the two S^2-valued Gauss maps of (S^2, k) we choose an orthogonal splitting of Minkowski space $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$ with projections $t: \mathbf{E}^3 + \mathbf{E}^- \to \mathbf{E}$ and $\pi_{\mathbf{E}}: \mathbf{E}^3 + \mathbf{E}^- \to \mathbf{E}^3$. Now let U_s^i, $i = F, P$, denote the two unique sections

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of the normal bundle that satisfy the following:

$$\text{(E1)}\quad \begin{aligned} &\text{(a)}\quad \langle U_s^i, U_s^i\rangle = 0, \quad i = F, P;\\ &\text{(b)}\quad t_*(U_s^i(x)) \text{ is of unit length for all } x \in S^2, \quad i = F, P;\\ &\text{(c)}\quad \langle U_s^F, \mathrm{FUT}\rangle < 0;\\ &\text{(d)}\quad dV(-, -, U_s^P, U_s^F) \equiv dA(-, -). \end{aligned}$$

(Here dA is the induced area form on S^2 and equivalence is in the sense of orientations. The subscript s refers to the choice of splitting.) Translating these sections to the origin in $\mathbf{M}^4$ yields two maps $U_s^i: S^2 \to S_i^2$, $i = F, P$, where,

$$\text{(E2)}\qquad S_F^2 = LC \cap \{t^{-1}(1)\}, \qquad S_P^2 = LC \cap \{t^{-1}(-1)\}.$$

Notice that each S_i^2, $i = F, P$, in (E2) may be viewed as a splitting dependent model for the space of null lines in $\mathbf{M}^4$ and that for any two splittings the corresponding spheres can be canonically identified. Thus we get two well-defined *S^2-valued Gauss maps*, and we say that the U_s^i, $i = F, P$, represent the Gauss maps of (S^2, k), $g^i: S^2 \to S_i^2$, relative to a splitting of $\mathbf{M}^4$.

We observe that the S_i^2, $i = F, P$, carry preferred orientations; that is, the orientations such that the null sections of the submanifold $S_i^2 \subset \mathbf{M}^4$, $i = F, P$, defined by (E1) project via $\pi_E: \mathbf{M}^4 \to \mathbf{E}^3$ to the outward normals of the unit sphere in E^3. What is not well defined independent of splitting is an area form on the 2-spheres S_i^2, $i = F, P$. However, upon fixing a splitting the normalization implicit in (E2) induces area forms dA_s^i on the S_i^2, $i = F, P$, with total are 4π. We define the *split curvatures* K_s^i, $i = F, P$, of (M, k) relative to the splitting s by $(U_s^i)^* dA_s^i = K_s^i dA$, $i = F, P$.

In Theorem 5 (E7) of [K1] we showed that for all, smoothly immersed space-like 2-spheres in $\mathbf{M}^4$ and splittings $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$ we have

$$8\pi \le \int_{S^2} |K_s^F| + |K_s^P|\, dA.$$

The result

Our result here is the following.

Theorem 1. *If $k: S^2 \to \mathbf{M}^4$ is a space-like knotted 2-sphere then for all splittings $\mathbf{M}^4 = \mathbf{E}^3 + \mathbf{E}^-$,*

$$16\pi \le \int_{S^2} |K_s^F| + |K_s^P|\, dA.$$

Proof. We will show that if for some splitting the inequality fails to hold then we can construct an embedded 3-disk whose boundary is the given embedded 2-sphere. Now, if the inequality fails to hold then the expected number of preimages of *both* U_s^F and $-U_s^P$ (viewed as maps $S^2 \to S_F^2$) is strictly less than four. Thus there exists a regular value $\eta^F \in S_F^2$ for both U_s^F and $-U_s^P$ such that the total number of preimages $\{(U_s^{F-1})^{-1}(\eta)^F\} \cup \{(-U_s^P)^{-1}(\eta)^F\}$ is three or less. Since such preimages are nondegenerate critical points for $\langle \eta^F, -\rangle \circ k: S^2 \to \mathbf{R}$, where $\langle \eta^F, -\rangle: \mathbf{M}^4 \to \mathbf{R}$ is a linear null height function, this total number of preimages must be exactly two, particularly, $\{x_{\min}, x_{\max}\} \in S^2$. Let $cv_\alpha = \langle \eta^F, x_\alpha\rangle$, $\alpha = \max, \min$, denote the critical values. Now choose $\eta^P \in$

S^2_P and consider the linear projection $\mathrm{proj}: \mathbf{M}^4 \to (\eta^P)^\perp = (\eta^F+\eta^P)^\perp + \eta^P = \mathbf{N}^3$ onto a null plane $\mathbf{N}^3$ with kernel η^F (see Figure 1a). Next consider the affine homotopy k_μ: $S^2 \times [0, 1] \to \mathbf{M}^4$, $k_0 = k$, $k_1(S^2) \subset \mathbf{N}^3$, obtained by pushing down along the fibers of proj to a fixed plane parallel to $(\eta^P)^\perp$, which we have identified with $\mathbf{N}^3$ (the image of $\mathrm{proj}: \mathbf{M}^4 \to \mathbf{N}^3$). Notice that since k_0 is space-like, each $k_\mu: S^2 \to \mathbf{M}^4$ is an immersion (i.e., the fibers of proj are transverse to k). It follows that $\langle \eta^4, -\rangle \circ k_\mu: S^2 \to \mathbf{R}$ is independent of $\mu \in [0, 1]$ (see Figure 1a).

On a sufficiently small neighborhood U of $x_{\min} \in S^2$, the map k_1 (and hence the total map $k_\mu | U \times [0, 1] \to \mathbf{M}^4$) is an embedding. This is because the fibers of proj are parallel lines and $x_{\min} \in S^2$ is the *unique* minimum point. It follows that for all $\mu \in [0, 1]$ the intersection $k_\mu(S^2) \cap \mathrm{proj}^{-1}(U)$ is a section of the line bundle $\mathrm{proj}^{-1}(U) \to U$. To see this, view $\mathrm{proj}^{-1}(k_1(S^2))$ as the image of an immersion $\tilde{k}: S^2 \times \mathbf{R} \to \mathbf{M}^4$. Then every k_μ factors as a section sec_μ of $S^2 \times \mathbf{R} \to S^2$.

$$
\begin{tikzcd}
U \times \mathbf{R} \arrow[r] & S^2 \times \mathbf{R} \arrow[r, "\tilde{k}"] \arrow[d] & \mathbf{M}^4 \arrow[d, "\mathrm{proj}"] \\
U \arrow[u, "\mathrm{sec}_\mu"'] \arrow[r] & S^2 \arrow[r, "k"'] & \mathbf{N}^3
\end{tikzcd}
$$

Since $k_\mu(U \times [0, 1])$ agrees with $\tilde{k} \circ \mathrm{sec}_\mu(U \times [0, 1])$, we have our claim.

Now by the Morse lemma there is a C^∞ closed embedded level curve $c: S^1 \to U$ that bounds a 2-disk in U. Thus $k_1 \circ c: S^1 \to \mathbf{N}^3$ is a closed C^∞-embedded curve that lies in an affine $\mathbf{E}^2 \subset \mathbf{N}^3$ that is parallel to $(\eta^F + \eta^P)^\perp$. Thus it is unknotted as a curve in $\mathbf{N}^3$. Further $k_\mu \circ c = \tilde{k} \circ \mathrm{sec}_\mu | S^1 \times [0, 1] \to \mathbf{M}^4$ has image that lies in the embedded cylinder $S^1 \times \mathbf{R} \simeq \mathrm{proj}^{-1}(k_1 \circ c(S^1)) \subset M^4$ (see Figure 1b). Since $k_1 \circ c(S^1)$ lies in a parallel of $(\eta^F + \eta^P)^\perp$, this embedded cylinder lies in an affine hyperplane parallel to $(\eta^F)^\perp$. It follows that for all $\mu \in [0, 1]$ the closed curve $k_\mu \circ c: S^1 \to S^1 \times \mathbf{R} \subset \mathbf{M}^4$ is a section of the

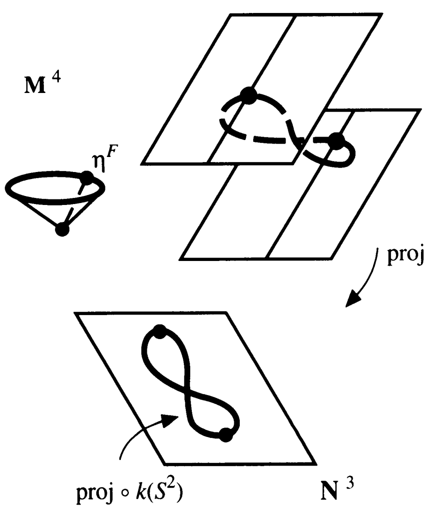

FIGURE 1a

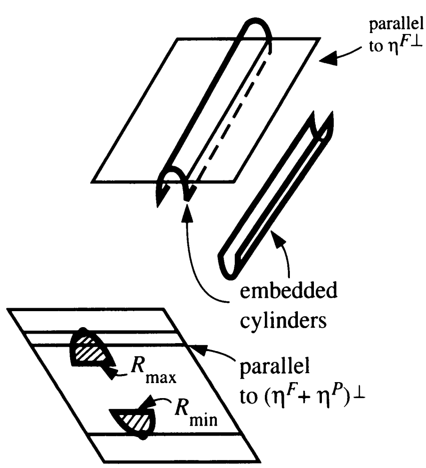

FIGURE 1b

embedded cylinder in the affine hyperplane and hence bounds a 2-disk. Thus $k_0 \circ c(S^1)$, a level curve in the given 2-sphere $k_0(S^2)$, bounds a C^0 2-disk in level hyperplane $\mathbf{N}^3 - \{x \in \mathbf{M}^4 | \langle \eta^F, x \rangle = cv_{\min} + \rho\}$, $0 < \rho \in \mathbf{R}$. We write $\gamma: D^2 \to \mathbf{N}^3 \subset \mathbf{M}^4$ to denote such a disk.

We may *summarize* the above as follows: for $\alpha = \min, \max$ there exist neighborhood U_α of $x_\alpha \in S^2$ containing embedded level curves $k_1 \circ c_\alpha: S^1 \to \mathbf{N}^3_\alpha \subset \mathbf{M}^4$ that are the boundaries of C^0-disks $\gamma_\alpha: D^2 \to \mathbf{N}^3_\alpha$ contained in level hyperplanes $\mathbf{N}^3_\alpha$ given by $\langle \eta^F, - \rangle = cv_{\min} + \rho$ or $cv_{\max} - \rho$, $0 < \rho \in \mathbf{R}$. We will refer to a subset of the form $\{x \in \mathbf{M}^4 | cv_{\min} + \rho \le \langle \eta^F, x \rangle \le cv_{\max} - \rho\} = \mathrm{SLAB}(\rho)$ as a *ρ-slab* and note that $0 < \rho$ sufficiently small the above implies that $\mathrm{proj}[(k(S^2)\text{-interior}(\mathrm{SLAB}(\rho))]$ has exactly two connected components that bound nonempty compact regions, R_α in $\mathbf{N}^3_\alpha$ (see Figure 1b). We note that given a C^0-section of $\mathrm{proj}^{-1}(\partial R_\alpha) \to \partial R_\alpha$, $\alpha = \min, \max$, it can be extended to a C^0-section of $\mathrm{proj}^{-1}(R_\alpha) \to R_\alpha$. For the remainder of the argument we fix $0 < \rho \in \mathbf{R}$ as above and choose $0 < \overline{\rho} < \rho$.

Now we construct a complete C^∞-vector field X on $\mathbf{M}^4$ such that:

(1) Its flow ψ_t leaves invariant the given 2-spheres $k(S^2)$ with source and sink at $x_{\min}, x_{\max} \in S^2$, respectively.

(2) There exist compact closed sets $D^4 \supset \widetilde{D}^4$ with nonempty interior such that: D^4 contains $k(S^2)$ and the support of X; the set $\widetilde{D}^4$ contains the intersection of $k(S^2)$ with the complement of $\mathrm{SLAB}(\overline{\rho})$; for all x in $\widetilde{D}^4 \cap \mathrm{SLAB}(\overline{\rho})$, we have $d/dt \langle \eta^F, \psi_t(x) \rangle |_0 = 1$ where $0 < \overline{\rho} < \rho$ are fixed as above.

This is done by choosing an $\mathbf{E}^4$-structure on $\mathbf{M}^4$ so that η^F, η^P, and $(\eta^F + \eta^P)^\perp$ are mutually $\mathbf{E}^4$-orthogonal with η^F and η^P of unit length. Let $\mathbf{E}^4\mathrm{grad}$ denote the gradient of $\langle \eta^4, - \rangle$ on $\mathbf{M}^4$ and $S^2\mathrm{grad}$ its $\mathbf{E}^4$-orthogonal projection on the tangent bundle to S^2. Next consider an $\mathbf{E}^4$-tubular neighborhood of $k(S^2)$, denoted $\mathrm{TUBE} \subset S^2 \times \mathbf{R}^2$, which we may identify with a neighborhood of the zero-section in the $\mathbf{E}^4$-normal bundle. Now the difference $\mathbf{E}^4\mathrm{grad} - S^2\mathrm{grad}$ may be viewed as a section of this bundle and hence as a vector field Y defined on the total space of $S^2 \times \mathbf{R}^2$ that is tangent to the fibration $S^2 \times \mathbf{R}^2 \to S^2$. (Just extend by parallel transport in each fiber.) Let $f: S^2 \times \mathbf{R}^2 \to \mathbf{R}$ be a smooth function satisfying $0 \le f \le 1$ with support contaned in $\mathrm{TUBE} \subset S^2 \times \mathbf{R}^2$ and $f = 1$ exactly on the zero-section. It follows that $Z = \mathbf{E}^4\mathrm{grad} - fY$ defines a smooth vector field on $\mathbf{E}^4$ that satisfies (1). On the boundary of a sufficiently large 3-Bail $B^3(x_{\min})$ in $\{\langle \eta^F, - \rangle = cv_{\min}\}$ centered at $x_{\min}$, the vector field Z agrees with $\mathbf{E}^4\mathrm{grad}$. Now use the $\mathbf{E}^4\mathrm{grad}$ flow to construct a closed cylinder $\mathbf{R} \times B^3(x_{\min})$ that intersects $\mathrm{SLAB}(\overline{\rho})$ in a closed set. The intersection of this cylinder with $\mathrm{SLAB}(0)$ defines $\widetilde{D}^4$ (see Figure 1c). Next set D^4 to be a round 4-ball in $\mathbf{E}^4$ that contains $k(S^2)$ and $\widetilde{D}^4$, and let $g: \mathbf{E}^4 \to \mathbf{R}$ be a smooth function with support in D^4 that agrees with $\|Z\|_{\mathbf{E}}^{-2}$ on $\widetilde{D}^4$. We define $X = gZ$ and (2) follows by construction.

Finally we construct the desired 3-disk with $k(S^2)$ as boundary. Start with the union $\mathrm{Cyl} = \{\bigcup \psi_t(\gamma_{\min}(D^2)) | t \in [0, cv_{\max}]\}$ where D^2 is the 2-disk constructed at the beginning of the proof. Since the flow is a diffeomorphism, this is an embedded C^0-cylinder $D^2 \times \mathbf{R}$ in $\mathbf{M}^4$ that intersects the boundary of $\mathrm{SLAB}(\rho)$ in a pair of embedded 2-disks in $\mathbf{N}^3_{\min}$ and $\mathbf{N}^3_{\max}$, respectively (see Figure 1d). We may use the $c_\alpha(S^1)$ with this pair of disks to define C^0-

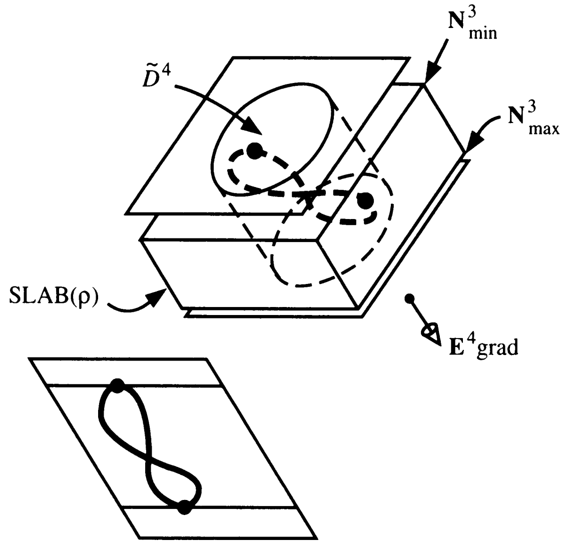

FIGURE 1c

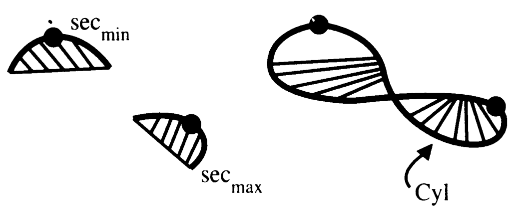

FIGURE 1d

sections of $\text{proj}^{-1}(\partial R_\alpha) \to \partial R_\alpha$, $\alpha = \min, \max$. Upon C^0-extending both of these sections to $\sec_\alpha: R_\alpha \to \text{proj}^{-1}(R_\alpha)$ we have that the union $\{\text{sect}_{\min}(R_{\min}) \cup \sec_{\max}(R_{\max})\}$ with $\{\text{Cyl} \cap \text{SLAB}(\rho)\}$ is the desired 3-disk, and we are finished.

EXAMPLES

The simplest examples of knotted 2-spheres are the spun knots of Artin [A]. Choose a transverse section for a rotation $\text{SO}(2, 0) \subset \text{SO}(3, 1)$; then there is an axis $\mathbf{M}^2 \subset \mathbf{M}^3$ and any curve $c: S^1 \to \mathbf{M}^3$ that nontrivially intersects the axis $\mathbf{M}^2$ so as to be invariant under $\mathbf{M}^3$ reflection thru the axis $\mathbf{M}^2$ yields a 2-sphere of revolution. If this profile curve is space-like then so is the 2-sphere of revolution. If the profile curve intersects the axis $\mathbf{M}^2$ in exactly 2-points, a and b, then Artin observed that $\pi_1(\text{2-sphere})^c = \pi_1(\text{profile} \cup \overline{ab})^c$. Thus in Figure 2 we have the profile of a knotted 2-sphere. Notice also that the Gauss images each cover their respective S_i^2, $i = F, P$, thrice with positive orientation. Thus the total split curvature, for the splitting implicit in Figure 2 (see the next page), is greater than 24π.

Corollary 2. *If both split curvatures of $k: S^2 \to \mathbf{M}^4$ are nonnegative then k is unknotted.*

Proof. We have $8\pi = \int_{S^2} |K_s^F| + |K_s^P|\, dA$.

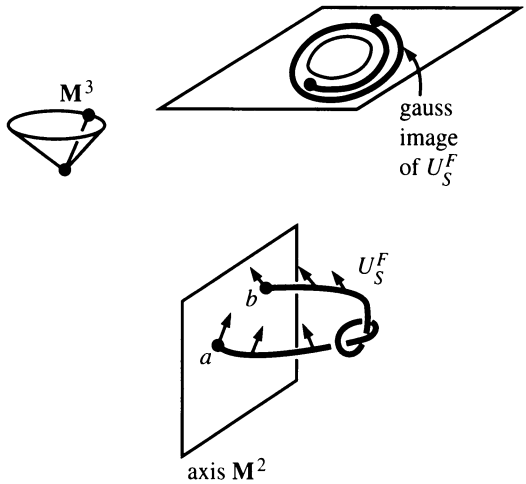

FIGURE 2

For an application of Theorem 1 see [K2]. We also note that the argument of Theorem 1 is valid in Minkowski 3-space. It follows that the total curvature of a knotted space-like loop in $\mathbf{M}^3$ is bounded below by 8π (see [K3]).

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