SPACES WHOSE $n$TH POWER IS WEAKLY INFINITE-DIMENSIONAL BUT WHOSE \((n+1)\)TH POWER IS NOT

ELŻBIETA POL

(Communicated by James E. West)

ABSTRACT. For every natural number $n$ we construct a metrizable separable space $Y$ such that $Y^n$ is weakly infinite-dimensional (moreover, is a C-space) but $Y^{n+1}$ is strongly infinite-dimensional.

1. Introduction

All our spaces are metrizable separable. Our terminology follows [3]. The symbol $P$ denotes the space of irrational numbers and $N$ the space of natural numbers. A space $X$ is weakly infinite-dimensional (w.i.d.) if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of $X$ there exist closed sets $L_1, L_2, \ldots$ such that $L_i$ is a partition between $A_i$ and $B_i$ and $\bigcap_{i=1}^{\infty} L_i = \emptyset$. A space is strongly infinite-dimensional (s.i.d.) if it is not w.i.d.

A space $X$ is called a C-space (or has property C) if for every sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers of $X$ there exists a sequence $\mathcal{Z}_1, \mathcal{Z}_2, \ldots$ of families of pairwise disjoint open subsets of $X$, the union of which covers $X$, such that each member of $\mathcal{Z}_i$ is contained in a member of $\mathcal{G}_i$ (see, for example, [3, §8] or [18] for this notion). Every countable-dimensional space has property C, and every space with property C is weakly infinite-dimensional.

In [14] an example was given showing that the square of a w.i.d. space may be s.i.d., and in [12] we have constructed a C-space whose product with a 0-dimensional space is s.i.d. (under the Continuum Hypothesis, the 0-dimensional factor may be the space of irrationals $P$).

In this note we show that for every $n$ the assumption that the $n$th power $Y^n$ of $Y$ is a C-space may not prevent $Y^{n+1}$ from being s.i.d. In fact, we show that each complete C-space which behaves regularly with respect to finite products, but is not hereditarily w.i.d., is a source of such examples.

Theorem 1. Let $X$ be a completely metrizable space such that

(i) for each C-space $E$, the product $X \times E$ is a C-space,
(ii) $X$ contains a strongly infinite-dimensional subspace.

Received by the editors July 5, 1991.

1991 Mathematics Subject Classification. Primary 54F45, 54B10.

Key words and phrases. Weakly infinite-dimensional, products, property C.
Then for each \( n \in \mathbb{N} \) there exists \( Y \subset P \times X \), \( P \) being the irrationals, such that

(a) \( Y^n \) is a C-space (hence is weakly infinite-dimensional),

(b) \( Y^{n+1} \) is strongly infinite-dimensional.

**Theorem 2.** Under the assumptions of Theorem 1 there exists \( Y \subset P \times X \) such that \( Y^n \) is a C-space for every \( n \in \mathbb{N} \) but the product of \( Y \) with a certain subspace of irrationals is strongly infinite-dimensional.

Note that the weakly infinite-dimensional compactum which is not countable-dimensional, constructed by R. Pol in [13], satisfies conditions (i) and (ii) (see [18, §3, first Corollary] for the proof of property (i)).

Garity, Hattori, Rohm, and Yamada (see [4, 5, 6, 18, 19]) have shown that in the presence of \( \sigma \)-compactness the products of spaces with property \( C \) behave in a regular way. Our examples, based on totally imperfect sets, lack compactness-like properties. Let us point out, however, that, as demonstrated in [15], under the Continuum Hypothesis even the Menger Property, a covering property close to \( \sigma \)-compactness, does not exclude the irregularities exhibited in Theorem 1.

The construction given in this paper improves our earlier construction [12, Example 1] by replacing the classical Bernstein’s sets with their modifications considered by van Douwen [2] and Przymusiński [16]. We decided to include, for the reader’s convenience, a simple construction of such sets in a form suitable for our purposes. As indicated by R. Pol (see [16, Remark 3]), this can be achieved by arguments due to Mycielski [11] and Kuratowski [8], concerning independent perfect sets.

Some basic ideas of the construction we apply go back to Michael [9, 10] (and were subsequently developed by Alster and Zenor [1], van Douwen [2], and Przymusiński [17]).

2. Auxiliary lemmas

The lemma below is a particular case of results of Kuratowski [8] and Mycielski [11] and the proof we give is an adaptation of their arguments to our special situation.

**Lemma 1.** Let \( \mathcal{F} \) be a countable family of continuous maps \( f : S \to T \) from a completely metrizable space \( S \) to a Hausdorff space \( T \). If \( S \) contains a subset \( A \) without isolated points such that each \( f \in \mathcal{F} \) is injective on \( A \), then \( S \) contains a perfect set with this property.

**Proof.** Let \( Z = \text{cl} \, A \) and let \( \mathcal{H}(Z) \) be the space of all compact subsets of \( Z \) with the Hausdorff metric. Let \( \mathcal{G} = \{ K \in \mathcal{H}(Z) : \text{all } f \in \mathcal{F} \text{ are injective on } K \} \). The set \( \mathcal{G} \) is a \( G_\delta \)-set in \( \mathcal{H}(Z) \) and, since the finite subsets of \( A \) form a subset of \( \mathcal{F} \) dense in \( \mathcal{H}(Z) \), \( \mathcal{G} \) is residual. Now, since the family of perfect subsets of \( Z \) is residual in \( \mathcal{H}(Z) \) (cf. [8, Proposition 2]), \( \mathcal{G} \) contains a perfect set.

The following result was essentially proved in [2, 16].

**Lemma 2.** Let \( \mathcal{F} \) be a countable family of continuous maps \( f : S \to T \) from a completely metrizable space \( S \) to a Hausdorff space \( T \). Then there exists a
countable collection $\mathcal{D}$ of subsets of $S$ such that

(a) the sets $\bigcup\{f(D) : f \in \mathcal{F}\}$, for $D \in \mathcal{D}$, are pairwise disjoint,
(b) if $D \subset U$ with $D \in \mathcal{D}$ and $U$ open in $S$, then for some countable $E \subset T$ we have $S\setminus U \subset \bigcup\{f^{-1}(E) : f \in \mathcal{F}\}$.

Proof. We repeat the classical Bernstein's argument. Let $\{F_\alpha : \alpha < 2^\omega\}$ be the collection of all perfect sets in $S$ on which every $f \in \mathcal{F}$ is injective (if this collection is empty, then put $\mathcal{D} = \{\emptyset, \emptyset, \ldots\}$). Let us choose inductively (with respect to pairs $(i, \alpha)$) points $x_i^\alpha \in F_\alpha$ with $f(x_i^\alpha) \neq g(x_i^\beta)$, if $(i, \alpha) \neq (j, \beta)$, where $i, j \in \mathbb{N}, \alpha, \beta < 2^\omega, f, g \in \mathcal{F}$, and let us set $D_i = \{x_i^\alpha : \alpha < 2^\omega\}, \mathcal{D} = \{D_i : i = 1, 2, \ldots\}$. Suppose that $D_i \subset U$, where $U$ is open in $S$, and let $M$ be a maximal subset of $S\setminus U$ such that each $f \in \mathcal{F}$ is injective on $M$. By Lemma 1, the set $M$ is dispersed, hence countable, and (b) follows with $E = \bigcup\{f(M) : f \in \mathcal{F}\}$.

The next lemma is due to Rubin [20]. It is not, however, stated explicitly in [20] and, instead of referring the reader to some details of Rubin's proof, we decided to indicate a standard argument needed to derive the fact we need from a theorem formulated in [20].

**Lemma 3.** Every strongly infinite-dimensional space $X$ contains a strongly infinite-dimensional totally disconnected subspace.

**Proof.** Indeed, as proved by Rubin [20, Theorem 3.1], every s.i.d. space $X$ contains an s.i.d. subspace $Y$ all of whose subspaces are either 0-dimensional or strongly infinite-dimensional. On the other hand, each space $Y$ of dimension $\geq 2$ contains a totally disconnected subspace of positive dimension. This can be justified as follows: let $A$ and $B$ be disjoint closed subsets of $Y$ such that each partition in $Y$ between $A$ and $B$ has positive dimension and let $f : Y \to [0, 1]$ be a continuous mapping with $f^{-1}(0) = A$ and $f^{-1}(1) = B$. By Hilgers' argument (see [7, Chapter II, §27, IX, Theorem 1]) there exists $M \subset Y$ with $f(M)$ irrationals (in fact, $M \cap f^{-1}(t)$ is a singleton for each irrational $t$) such that each $G_\delta$-set containing $M$ contains also some $f^{-1}(t)$, where $t \in (0, 1)$. Then $M$ is totally disconnected, but by the enlargement theorem (see [7, Chapter II, §27, IV, Theorem 1]) $M$ is not 0-dimensional as $\dim f^{-1}(t) \geq 1$ for each $t \in (0, 1)$.

### 3. Proofs

**Proof of Theorem 1** (see [3] for facts about weakly infinite-dimensional spaces). Suppose that $X$ is a space satisfying conditions (i) and (ii) of Theorem 1. Set $Z = P \times X$ and let $p : Z \to P$ be the projection. For $j \leq m$ let $p_j : Z^m \to P$ be the composition of the $j$th projection with $p$. Put $S = \bigoplus_{m=1}^\infty Z^m$ and let $f_j : S \to P$ be such that $f_j(x)$ is $p_j(x)$ if $x \in Z^m$ with $m \geq j$ or $p_1(x)$ otherwise.

Let $D_1, D_2, \ldots$ be the sets satisfying conditions (a) and (b) of Lemma 2 for $S$, $T = P$, and $\mathcal{F} = \{f_1, f_2, \ldots\}$. Then for every $i \in \mathbb{N}$ the sets $B_i = \bigcup_{j=1}^\infty f_j(D_i)$ are disjoint totally imperfect subsets of $P$ such that if $A_i = \bigcup_{i=1}^\infty B_i$
$B_i \times X \subset Z$, then

if $m \in \mathbb{N}$ and $U \subset Z^m$ is open and contains $A_i^m$ then

$Z^m \setminus U$ is contained in countably many fibers $p_j^{-1}(d)$,

where $d \in P$ and $j \leq m$.

This follows directly from conditions (a) and (b) of Lemma 2.

Since the space $X$ is not hereditarily w.i.d., by Lemma 3 there exists an s.i.d. subset $M$ of $X$ which is totally disconnected. Let $h : M \to P$ be an injection (see [7, Chapter V, §46, V, Theorem 3]); then the set

$$\tilde{M} = \{(h(x), x) : x \in M\} \subset Z$$

is homeomorphic to $M$.

Let us fix $n \in \mathbb{N}$. For every $i \leq n + 1$ let $C_i = \bigcup \{A_j : j \neq i, j \leq n + 1\}$ and let $Y_i = C_i \cup \tilde{M}$ be the subspace of $Z$.

We will show that the space $Y = \bigoplus_{i=1}^{n+1} Y_i$ satisfies conditions (a) and (b) of Theorem 1.

To prove (a) it suffices to show that for every $k \leq n$ and every $i_1, \ldots, i_k \in \{1, \ldots, n + 1\}$

$$\text{(2)} \quad \text{the Cartesian product } Y_{i_1} \times \cdots \times Y_{i_k} \text{ has property } C.$$ 

Suppose that $m = 1$ or (2) is true for $k < m$, where $m \leq n$. Put $K = Y_{i_1} \times \cdots \times Y_{i_m}$. To prove that $K$ is a $C$-space it suffices to show that

$$\text{if } G_1, G_2, \ldots \text{ is a sequence of open covers of } K \text{ then there exists a pairwise disjoint open refinement } \mathcal{G}_n \text{ of } G_n \text{ such that}$$

$$\text{(3)} \quad \bigcup \{G_n : n \in \mathbb{N}\} \text{ covers } K.$$ 

Since $m \leq n$, there exists $\alpha \in \{1, \ldots, n + 1\} \setminus \{i_1, \ldots, i_m\}$. We have $A_j \subset C_i$ for $l = 1, \ldots, m$, thus $A_i^m \subset K$. Since $A_i^m$ is the Cartesian product of $X^m$ with a subset $B_j$ of $P^m$ and since $X$ satisfies (i), the space $A_i^m$ is a $C$-space. Thus, for every $p = 1, 2, \ldots$ there exists a family $\mathcal{W}_{2p-1}$ consisting of disjoint open subsets of $Z^m$ such that $\mathcal{W}_{2p-1} = \{W \cap K : W \in \mathcal{W}_{2p-1}\}$ is a refinement of $\mathcal{G}_{2p-1}$ and the set $U = \bigcup_{p=1}^{\infty} \mathcal{W}_{2p-1}$ contains $A_i^m$. By (1), the set $Z^m \setminus U$ is contained in countably many fibers of the form $H = \{x \in Z^m : p_k(x) = \alpha\}$, where $k \leq m$ and $d \in D \subset P$, with $|D| \leq \aleph_0$. If $m > 1$ then for every such $H$ the set $H \cap K$ is closed in $K$ and is homeomorphic with $\mathbb{P}_{l \neq k} Y_{i_l} \times X$, or with $\mathbb{P}_{i \neq k} Y_{i_1} \times \{d, h^{-1}(\alpha)\}$, if $d \in h(M) \setminus (\bigcup \{B_j : j \neq i_k, j \leq n + 1\})$, or it is empty. By the inductive assumption and (i) the set $H \cap K$ is a $C$-space. If $m = 1$, then $H \cap K$ is homeomorphic to $X$, one-point, or the empty space, hence it is a $C$-space. In both cases the set $K \setminus U$ is contained in countably many closed subsets of $K$ of the form $H \cap K$ with property $C$, hence it has property $C$. Thus for every $p = 1, 2, \ldots$ there exists a pairwise disjoint open refinement $\mathcal{W}_p$ of $\mathcal{G}_p$ such that $\bigcup_{p=1}^{\infty} \mathcal{W}_p$ covers $K \setminus U = K \setminus \bigcup_{p=1}^{\infty} \mathcal{W}_{2p-1}$. The families $\mathcal{G}_1, \mathcal{G}_2, \ldots$ satisfy the required conditions, which finishes the proof of (3).

We will show now that $Y^{n+1}$ is strongly infinite-dimensional. It suffices to prove that $Y_1 \times Y_2 \times \cdots Y_{n+1}$ is s.i.d. Let

$$\Delta = \{x = (x_1, \ldots, x_{n+1}) \in Y_1 \times \cdots \times Y_{n+1} : x_1 = \cdots = x_{n+1}\}.$$
Let \( x = (z, z, \ldots, z) \in \Delta \). If \( z \notin \widetilde{M} \), then \( z \in A_j \) for some \( j \leq n + 1 \) and at the same time \( z \in Y_j \setminus M \subset C_j \), contrary to \( C_j \cap A_j = \emptyset \). Thus \( z \in \widetilde{M} \), which implies that \( \Delta \) is homeomorphic with \( \widetilde{M} \). Thus \( Y_1 \times \cdots \times Y_{n+1} \) is s.i.d., since it contains a closed subspace which is s.i.d.

**Proof of Theorem 2.** Let \( B_1, B_2, \ldots \) be the sets constructed in the proof of Theorem 1. Then the sets \( B'_1 = B_1 \) and \( B'_2 = P \setminus B'_1 \) form a decomposition of \( P \) into two disjoint Bernstein sets such that condition (1) is satisfied for \( A_i = B'_i \times X \), \( i = 1, 2 \). Let \( M, h : M \to P \), and \( \widetilde{M} \) be as in the proof of Theorem 1 and let 

\[
\widetilde{M}_i = \{(h(x), x) : x \in h^{-1}(B'_i)\}
\]

for \( i = 1, 2 \). Since \( \widetilde{M} = \widetilde{M}_1 \cup \widetilde{M}_2 \) is s.i.d., then either \( \widetilde{M}_1 \) or \( \widetilde{M}_2 \) is s.i.d. Suppose that \( \widetilde{M}_1 \) is s.i.d. and let \( Y = M_1 \cup A_2 \).

We will show that

\[
(4) \quad \text{the product } Y \times B'_1 \text{ of } Y \text{ with a subspace } B'_1 \text{ of } P \text{ is s.i.d.}
\]

Similarly as in [12, Example 1], let \( f : Y \to P \) be the projection; then \( f^{-1}(B'_1) = \widetilde{M}_1 \) and \( \text{Graph}(f|\widetilde{M}_1) \) is homeomorphic to \( \widetilde{M}_1 \), so it is s.i.d. This implies that \( Y \times B'_1 \) is s.i.d. (see [12, Lemma 2]).

It remains to check that

\[
(5) \quad Y^n \text{ is a } C\text{-space for every } n \in N.
\]

This can be proved inductively, similarly to condition (2), using the facts that \( Y^n \supset A_2^n \) for every \( n \in N \) and \( A_2 \) satisfies (1).

**References**

15. ——, Note on products of weakly infinite-dimensional spaces with Menger Property, preprint.

Department of Mathematics, Warsaw University, ul. Banacha 2, 00-913 Warszawa 59, Poland
E-mail address: POL@MIMUW.EDU.PL