THE iterated weak hilbert procedure

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abstract. applying pisier's concept of weak properties to weak hilbert spaces we obtain the so-called weak weak hilbert spaces. our main result is that the classes of weak hilbert spaces and of weak weak hilbert spaces coincide. on the other hand we show that a generalization for operators does not hold.

introduction

in the following we introduce a general concept of how to produce weak ideals, in particular, those of weak weak hilbert operators. for this let \((A, \alpha), (B, \beta)\) be quasi-banach ideals, \(X, Y\) banach spaces, and \(T : X \to Y\) an operator. we say that \(T \in Q(\alpha, \beta)\) [the quotient of \((A, \alpha)\) and \((B, \beta)\)] if there is a constant \(c \geq 0\) such that for all operators \(u \in A(l_2, X)\)

\[\beta(Tu) \leq c\alpha(u).\]  

from this we deduce easily by trace duality that for all operators \(u \in A(l_2, X), v \in B^*(Y, l_2)\),

\[\sum_{k \in \mathbb{N}} a_k(v Tu) \leq c\alpha(u)\beta^*(v),\]  

where \(a_k\) denotes the \(k\)th approximation number and \((B^*, \beta^*)\) the conjugate ideal of \((B, \beta)\). for more precise definitions see the preliminaries below.

by pisier's [pi2] concept of weak properties we say that \(T \in W(\alpha, \beta)\) (the weak ideal of \((A, \alpha)\) and \((B, \beta)\)) if there is a constant \(c \geq 0\) such that for all operators \(u \in A(l_2, X), v \in B^*(Y, l_2)\),

\[\sup_{k \in \mathbb{N}} \|k a_k(v Tu)\| \leq c\alpha(u)\beta^*(v).\]  

observe that \((W)\) is obtained by replacing in \((Q^*)\) the \(l_1\)-norm by the weaker \(l_{1, \infty}\)-norm. if we define \(q(\alpha, \beta)(T) = \inf c, w(\alpha, \beta)(T) = \inf c,\) where the infimum is taken over all \(c\) such that \((Q), (W)\) holds, respectively, then \((Q(\alpha, \beta), q(\alpha, \beta)), (W(\alpha, \beta), w(\alpha, \beta))\) are quasi-banach ideals and

\[Q(\alpha, \beta) \subset W(\alpha, \beta), \quad w(\alpha, \beta) \leq q(\alpha, \beta).\]  

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In this paper our aim is to apply this general scheme to the ideal norms 
\( \alpha = \pi_2^d \) and \( \beta = \pi_2 \), where \( (\Pi_2, \pi_2) \) denotes the Banach ideal of absolutely 
2-summing operators and \( (\Pi_2^d, \pi_2^d) \) the dual Banach ideal of \( (\Pi_2, \pi_2) \) (i.e., 
\( \pi_2^d(u) = \pi_2(u^*) \)). Then by a well-known result of Kwapien [K] \( T \in Q(\pi_2^d, \pi_2) \)  
iff \( T \) is a Hilbert operator (operators, that factor through a Hilbert space).  
Applying (W) leads to Pisier’s [Pi1] definition of weak Hilbert operators, hence  
\( T \) is a weak Hilbert operator if there is a constant \( c \geq 0 \) such that for all operators  
\( u \in \Pi_2^d(l_2, X), \ v \in \Pi_2(Y, l_2), \)
\[
\sup_{k \in \mathbb{N}} k a_k(v Tu) \leq c \pi_2(u^*) \pi_2(v)
\]
(recall that \( \pi_2^* = \pi_2 \)). A result of Pisier [Pi1] (see also [DJ]) says that \( T \) is a  
weak Hilbert operator iff there is a constant \( c \geq 0 \) such that for all operators  
\( u \in \Pi_2^d(l_2, X), \)
\[
\sup_{k \in \mathbb{N}} k^{1/2} a_k(Tu) \leq c \pi_2(u^*);
\]
but this means that \( T \) is a weak Hilbert operator iff \( T \in Q(\pi_2^d, l_2^{q, \infty}) \), where  
\( (\mathcal{L}_2^{q, \infty}, l_2^{q, \infty}) \) is the quasi-Banach ideal such that the quasi-ideal norm  
\[
l_2^{q, \infty}(u) = \sup_{k \in \mathbb{N}} k^{1/2} a_k(u) < \infty.
\]
Hence we are in the situation to reapply (W) to \( \alpha = \pi_2^d \) and \( \beta = l_2^{q, \infty} \). This  
leads to the definition of the weak weak Hilbert operators.  
\( T \) is a \textit{weak weak Hilbert operator} if there is a constant \( c \geq 0 \) such that for all operators  
\( u \in \Pi_2^d(l_2, X), \ v \in (\mathcal{L}_2^{q, \infty})^*(Y, l_2), \)
\[
\sup_{k \in \mathbb{N}} k a_k(v Tu) \leq c \pi_2(u^*)(l_2^{q, \infty})^*(v).
\]
A Banach space \( X \) is called weak Hilbert space, \textit{weak weak Hilbert space} if \( \text{id}_X \)  
is a weak Hilbert operator, weak weak Hilbert operator, respectively. Our main  
results are the following:

1. **Theorem.** The classes of weak Hilbert spaces and weak weak Hilbert spaces coincide.

2. **Theorem.** There is a weak weak Hilbert operator, which is not a weak Hilbert operator.

**Preliminaries**

We use standard Banach space notations. In particular, we have for all Banach spaces \( X \) and subspaces \( E \subset X : i_E : E \to X, \ x \mapsto x \). The Lorentz  
sequence spaces \( (l_p,q, || \cdot ||_{p,q}), (l_{p,q}^n, || \cdot ||_{p,q}), n \in \mathbb{N}, 0 < p, q \leq \infty \), are  
defined in the usual way. Standard references on \( s \)-numbers and operator ideals  
are the monographs of Pietsch [P1, P2]. The ideal of all linear bounded,  
finite-dimensional operators are denoted by \( \mathcal{L}, \mathcal{F} \), respectively.

Let \( (A, \alpha) \) be a quasi-Banach ideal. The component \( A^*(X, Y) \) of the  
conjugate ideal \( (A^*, \alpha^*) \) is the class of all operators \( T \in \mathcal{L}(X, Y) \) such that  
\[
\alpha^*(T) = \sup\{ |\text{tr} \ TS| ||S \in \mathcal{F}(Y, X), \ \alpha(S) \leq 1 \} < \infty.
\]
The component \( A^d(X, Y) \) of the dual ideal \((A^d, \alpha^d)\) is the class of all operators \( T \in \mathcal{L}(X, Y) \) such that \( T^* \in A(Y^*, X^*) \). We set
\[
\alpha^d(T) = \alpha(T^*).
\]

We call a quasi-Banach ideal \((A, \alpha)\) injective if we have for all operators \( T \in \mathcal{L}(X, Y) \) and isometries \( I \in \mathcal{L}(X, Z) \) with \( IT \in A(X, Z) \) that \( T \in A(X, Y) \) and \( \alpha(T) = \alpha(IT) \).

Note that for all two quasi-Banach ideals \((A, \alpha), (B, \beta)\) the inclusion \( A \subset B \) implies \( \beta \leq c\alpha \) for some constant \( c \geq 0 \).

Next we recall the usual notation of some \( s \)-numbers of an operator \( T \in \mathcal{L}(X, Y) \).

\[
an(T) = \inf \{ \| T - S \| : S \in \mathcal{L}(X, Y), \ \text{rank}(S) < n \}
\]
the \( n \)th approximation number,

\[
x_n(T) = \sup \{ a_n(Tu) : u \in \mathcal{L}(l_2, X), \ \| u \| \leq 1 \}
\]
the \( n \)th Weyl number,

\[
h_n(T) = \sup \{ a_n(vTu) : u \in \mathcal{L}(l_2, X), \ \| u \| \leq 1, \ v \in \mathcal{L}(Y, l_2), \ \| v \| \leq 1 \}
\]
the \( n \)th Hilbert number.

If \( X \) is a Hilbert space then \( a_n(T) = x_n(T) \) and \( a_n(T) = a_n(T^*) \). The Weyl numbers are injective, i.e., if \( S \in \mathcal{L}(Y, Z) \) is an isometric then \( x_n(T) = x_n(IT) \). Furthermore they are multiplicative, i.e., for all operators \( S \in \mathcal{L}(Y, Z) \) and \( n, m \in \mathbb{N}, x_{m+n-1}(ST) \leq x_n(S)x_m(T) \). Recall that the Hilbert numbers are completely symmetric, i.e., \( h_n(T) = h_n(T^*) \).

Now let \( s \) be any \( s \)-number. The component \( \mathcal{L}_{p,q}^{s}(X, Y) \) of the quasi-Banach ideal \((\mathcal{L}_{p,q}, l_{p,q})\) is the class of all operators \( T \in \mathcal{L}(X, Y) \) such that
\[
l_{p,q}^s(T) = \| (s_k(T))_{k=1}^n \|_{p,q} < \infty.
\]
Since the Weyl numbers are injective, it is obvious that the Weyl ideal \((\mathcal{L}_{p,q}^s, l_{p,q}^s)\) is injective.

An operator is called a Hilbert operator \((T \in \Gamma_2(X, Y))\) if there is a Hilbert space \( H \) and operators \( S \in \mathcal{L}(X, H), \ R \in \mathcal{L}(H, Y) \) such that \( T = RS \). We denote by \( \gamma_2(T) = \inf \| R \| \| S \| \), where the infimum is taken over all such factorizations through a Hilbert space.

An operator is said to be absolutely 2-summing \((T \in \Pi_2(X, Y))\) if there is a constant \( c \geq 0 \) such that for all \( n \in \mathbb{N}, \ (x_k)_{k=1}^n \subset X \),
\[
\left( \sum_{k=1}^n \| Tx_k \|^2 \right)^{1/2} \leq c \sup_{x^* \in H^*} \left( \sum_{k=1}^n |(x_k, x^*)|^2 \right)^{1/2}.
\]

We let \( \pi_2(T) = \inf c \), where the infimum is taken over all \( c \) such that the inequality holds. Recall that \( \Pi_2 = \Pi_2^* \) with equal norms [P2].

\((\Gamma_2, \gamma_2)\) and \((\Pi_2, \pi_2)\) are Banach ideals [P2].

At the end of the preliminaries we want to mention those known results that we use essentially in this paper.

1. [P1] Let \( H \) be a Hilbert space, \( T \in \mathcal{L}(H, Y) \), and \( n \in \mathbb{N} \). If \( a_n(T) > 0 \), then for every \( \varepsilon > 0 \) there is an orthogonal family \( (x_k)_{k=1}^n \subset H \) such that \( a_k(T) \leq (1 + \varepsilon)\| Tx_k \| \) for all \( k = 1, \ldots, n \).
(2) [P2] Let $T \in \mathcal{L}(X, Y)$, $n \in \mathbb{N}$, and $\varepsilon > 0$. Then there are operators $S \in \mathcal{L}(l^n_2, X)$, $R \in \mathcal{L}(Y, l^n_2)$ such that $\|R\| \leq 1$, $\|S\| \leq (1 + \varepsilon)/h_n(T)$, and $\text{id}_{l^n_2} = RTS$.

(3) [Jo, Pi2, T-J] Let $n \in \mathbb{N}$ and $E$ be a Banach space with $\dim E = n$. Then there is an invertible operator $u \in \mathcal{L}(l^n_2, E)$ such that $\|u\| = 1$ and $\pi_2(u^{-1}) = n^{1/2}$.

(4) [P1] $\Pi_2 \subseteq \mathcal{L}_{2,\infty}^x$ and $l^n_{2,\infty} \leq \pi_2$.

(5) [DJ] Let $(A, \alpha)$ be an injective Banach ideal and $(B, \beta)$ an injective quasi-Banach ideal. Then $W(a^{\alpha \beta}, \beta) = W(\beta^{\alpha \beta}, \alpha^d)$.

(6) [DJ] Let $(A, \alpha)$, $(C, \gamma)$ be quasi-Banach ideals and $(f(n))_n$ be a positive sequence such that $c = \sup_{n \in \mathbb{N}}(\prod_{k=1}^{n} f(k)/f(n))^{1/n} < \infty$. Then we have for all $T \in \mathcal{L}(X, Y)$ the implication (i) $\rightarrow$ (ii) and the estimate $M_2(T) \leq M_1(T)$.

(i) There is a constant $M_1(T) > 0$ such that for all $u \in A(l^n_2, X)$, $v \in \Gamma(Y, l^n_2)$, $\sup_{k \in \mathbb{N}} f(k)ak(vTu) \leq M_1(T) \alpha(u) \gamma(v)$.

(ii) There is a constant $M(T) > 0$ such that for all $u \in A(l^n_2, X)$,

$$\sup_{k \in \mathbb{N}} \frac{f(k)}{\gamma(id: l^n_{2,\infty} \rightarrow l^n_{2,\infty})} a_k(Tu) \leq M_2(T) \alpha(u).$$

(7) [K] $\Gamma_2 = Q(\pi_2^d, \pi_2)$ with equal norms.

(8) [Pi2] A Banach space $X$ is a weak Hilbert space (id$_X \in W(\pi_2^d, \pi_2)$) iff there is a constant $c \geq 0$ and a number $0 < \delta \leq 1$ such that for all $n \in \mathbb{N}$ with $\delta n \geq 1$ and subspaces $E \subseteq X$ with $\dim E = n$ there is a subspace $F \subseteq E$ with $\dim F = \lfloor \delta n \rfloor$ and a projection $P_F \in \mathcal{L}(X, F)$ onto $F$ with $\gamma_2(P_F) \leq c$.

To prove Theorem 1 we start with

3. Proposition. Let $X$ be a Banach space and $c > 0$ a constant such that for all operators $u \in (\mathcal{L}_{2,\infty}^a)^{\ast d}(l^n_2, X)$, $v \in \Pi_2(X, l^n_2)$,

$$\sup_{k \in \mathbb{N}} ka_k(\gamma) \leq c(l^n_{2,\infty} \ast (u^{\ast})) \pi_2(v).$$

Then $X$ is a weak Hilbert space.

Proof. Since $\pi_2(id: l^n_{2,\infty} \rightarrow l^n_{2}) = k^{1/2}$, we deduce from the assumption, (6), and the multiplicity of the Weyl numbers that for all $w \in (l^n_{2,\infty} \ast d)(l^n_2, X)$,

$$l^n_{2,\infty}(w) = \sup_{k \in \mathbb{N}} k^{1/2} a_k(w) \leq ec(l^n_{2,\infty})^{\ast}(w^{\ast})$$

$$\leq ec \sup \left\{ \sum_{k \in \mathbb{N}} a_k(w^{\ast}u) \mid u \in \mathcal{F}(l^n_2, X^{\ast}), \ l^n_{2,\infty}(u) \leq 1 \right\}$$

$$\leq 2ec \sum_{k \in \mathbb{N}} \frac{x_k(w^{\ast})}{k^{1/2}} = 2ec \sum_{k \in \mathbb{N}} \frac{h_k(w)}{k^{1/2}}.$$ 

Now choose $0 < \delta \leq 1$ such that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{\lfloor \delta n \rfloor} \frac{1}{k^{1/2}} \leq \frac{1}{8ec} n^{1/2}.$$
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Fix \( n \in \mathbb{N} \) with \( \delta n \geq 1 \) and a subspace \( E \subset X \) with \( \dim E = \lfloor \delta n \rfloor \). By (3) there is an invertible operator \( u \in \mathcal{L}(l_2^n, E) \) with \( \|u\| = 1 \) and \( \pi_2(u^{-1}) \leq n^{1/2} \). Hence we obtain, with \( m = (n + 1)/2 \geq n/2 \), the multiplicity and the injectivity of the Weyl numbers and, by (4),

\[
\begin{align*}
\frac{n}{2} & \leq \sqrt{2}m^{1/2}a_n(\text{id}_n) \leq \sqrt{2}m^{1/2}x_m(u)x_m(u^{-1}) \\
& \leq \sqrt{2l_2^{\infty}(u)} \pi_2(u^{-1}) \frac{m^{1/2}}{m} \leq \sqrt{2l_2^{\infty}(u)} \left( \frac{n}{m} \right)^{1/2} \\
& \leq 2l_2^{\infty}(u) \leq 2l_2^{\infty}(i_Eu) \leq 4\epsilon c l_{\delta n}(i_Eu) \\
& \leq 4\epsilon c \left( \sum_{k=1}^{\lfloor \delta n \rfloor} \frac{h_k(i_Eu)}{k^{1/2}} + \sum_{k=\lfloor \delta n \rfloor}^n \frac{h_k(i_Eu)}{k^{1/2}} \right) \\
& \leq 4\epsilon c \left( \|i_Eu\| \frac{1}{8\epsilon c} n^{1/2} + \|u\|h_{\lfloor \delta n \rfloor}(i_Eu)2n^{1/2} \right) \\
& \leq \frac{1}{2}n^{1/2} + 8\epsilon c h_{\lfloor \delta n \rfloor}(i_E)n^{1/2};
\end{align*}
\]

but this implies \( h_{\lfloor \delta n \rfloor}(i_E) \geq 1/16\epsilon c \).

Let \( \epsilon > 0 \). Then by (2) there are operators \( R \in \mathcal{L}(X, l_2^{\lfloor \delta n \rfloor}) \) and \( S \in \mathcal{L}(l_2^{\lfloor \delta n \rfloor}, X) \) such that \( \|R\| \leq 1, \|S\| \leq (1 + \epsilon)/h_{\lfloor \delta n \rfloor}(i_E) \), and \( R_iS = \text{id}_{l_2^{\lfloor \delta n \rfloor}} \).

Then we define \( F = S(l_2^{\lfloor \delta n \rfloor}) \subset E, \tilde{S} : l_2^{\lfloor \delta n \rfloor} \to F, x \mapsto S(x) \in \mathcal{L}(l_2^{\lfloor \delta n \rfloor}, F) \), and \( PF = \tilde{S}R \in \mathcal{L}(X, F) \). Hence \( \dim F = \lfloor \delta n \rfloor, PF \) is a projection onto \( F \), and

\[
\gamma_2(P_F) \leq \|R\| \|\tilde{S}\| \leq \frac{(1 + \epsilon)}{h_{\lfloor \delta n \rfloor}(i_E)} \leq (1 + \epsilon)16\epsilon c.
\]

Therefore (8) yields our assertion. \( \Box \)

Now we are able to prove Theorem 1.

Proof of Theorem 1. Assume that \( X \) is a weak weak Hilbert space. This means \( \text{id}_X \in W(\pi_2^d, l_2^{\infty}) = W(\pi_2^d, l_2^{\infty}) \). Since \( (\Pi_2, \pi_2) \) is an injective Banach ideal and \( (\mathcal{L}^{\infty}_{2, \infty}, l_2^{\infty}) \) is an injective quasi-Banach ideal. (5) implies that \( \text{id}_X \in (W((l_2^\infty)^d, \pi_2)^d) \); hence we deduce from Proposition 3 that \( X^* \) is a weak Hilbert space. Then by Pisier [Pil] (or again (5)) it is well known that \( X \) is a weak Hilbert space. \( \Box \)

Remark. In fact, we proved that the assumption of Proposition 3 characterizes weak Hilbert space.

To prove Theorem 2 we need the following.

4. Lemma. Let \( X \) be an Banach space and \( v \in (\mathcal{L}^{\infty}_{2, \infty})^*(X, l_2) \). Then

\[
\sum_{k\in\mathbb{N}} \frac{a_k(v)}{k} \leq \sqrt{2(l_2^{\infty})^*(v)}.
\]

Proof. Let \( n \in \mathbb{N} \) such that \( a_n(v) = a_n(v^*) > 0 \). Then by (1) for \( \epsilon > 0 \) we find a complete orthonormal system \( (e_k)_{k\in\mathbb{N}} \subset l_2 \) and a sequence \( (x_k)_{k=1}^n \subset B_X \) such that for all \( k = 1, \ldots, n \),

\[
a_k(v) = a_k(v^*) \leq (1 + \epsilon)\|v^*(e_k)\| \leq (1 + \epsilon)^2 \langle x_k, v^*(e_k) \rangle.
\]
Let us define \( u = \sum_{k=1}^{n} e_k \otimes x_k/k \in \mathcal{L}(l_2, X) \) and let \( P_k \in \mathcal{L}(l_2, l_2) \) be a projection onto \( \text{span}\{e_i\}_{i=1}^{k} \) for \( k = 1, \ldots, n \) (\( \text{span}\{e_i\}_{i=1}^{0} = \{0\} \)). Then we have

\[
a_k(u) \leq \|u - uP_k\| = \|u|_{\text{span}\{e_i\}_{i=k}}\| = \sup\left\{ \left\| \sum_{i=k}^{\infty} \alpha_i e_i \right\| \right\| \sum_{i=k}^{\infty} |\alpha_i|^2 \leq 1 \right\}
\]

\[
\leq \left( \sum_{i=k}^{\infty} \frac{\|x_i\|^2}{i^2} \right)^{1/2} \leq \sqrt{2} \frac{1}{k^2}.
\]

Hence this implies

\[
\sum_{k=1}^{n} \frac{a_k(v)}{k} \leq (1 + \varepsilon)^2 \sum_{k=1}^{n} \frac{1}{k} \langle x_k, v^*(e_k) \rangle
\]

\[
\leq (1 + \varepsilon)^2 \sum_{k=1}^{n} \langle u(e_k), v^*(e_k) \rangle = (1 + \varepsilon)^2 \text{tr} uv
\]

\[
\leq (1 + \varepsilon)^2 |l_2, \infty(u)(l_2, \infty)^*(v)
\]

\[
= (1 + \varepsilon)^2 \sup_{k=1,\ldots,n} k^{1/2} a_k(u)(l_2, \infty)^*(v)
\]

\[
\leq \sqrt{2}(1 + \varepsilon)^2 (l_2, \infty)^*(v).
\]

Since \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) are arbitrary, the assertion is proved. \( \Box \)

5. **Proposition.** For every \( n \in \mathbb{N} \) and every operator \( T \in \mathcal{L} \) with rank\( (T) = n \) we have

\[
w(\pi_2^d, l_2, l_2, \infty)(T) \leq \frac{8}{1 + \ln n} \frac{n^{1/2}}{\|T\|}.
\]

**Proof.** Let \( u \in \Pi_2^d(l_2, X) \) and \( v \in (\mathcal{L}_2^d, l_2, \infty)^*(Y, l_2) \). Then the multiplicity of the Weyl numbers, (4), and Lemma 4 imply

\[
\sup_{k \in \mathbb{N}} k a_k(v Tu) = \sup_{k \in \mathbb{N}} k \pi_k(u^* T^* v^*) \leq 2l_2, \infty(u^*)l_2, \infty(T^* v^*)
\]

\[
\leq 2\pi_2(u^*) \sup_{k=1,\ldots,n} k^{1/2} \pi_k(T^* v^*)
\]

\[
\leq 2\pi_2(u^*) \|T\| \sup_{k=1,\ldots,n} k^{1/2} (1 + \ln k) a_k(v)
\]

\[
\leq \frac{4}{\sqrt{e}} \frac{n^{1/2}}{1 + \ln n} \|T\| \pi_2(u^*) \sup_{k=1,\ldots,n} (1 + \ln k) a_k(v)
\]

\[
\leq \frac{8}{\sqrt{e}} \frac{n^{1/2}}{1 + \ln n} \|T\| \pi_2(u^*) \sum_{k \in \mathbb{N}} \frac{a_k(v)}{k}
\]

\[
\leq 8 \frac{n^{1/2}}{1 + \ln n} \|T\| \pi_2(u^*)(l_2, \infty)^*(v).
\]

Hence the definition of the weak weak Hilbert norm completes the proof. \( \Box \)

An immediate consequence is
Proof of Theorem 2. Let \( i_{2,\infty}^n : l_2^n \to l_\infty^n \) and \( i_{\infty,2}^n : l_\infty^n \to l_2^n \) for \( n \in \mathbb{N} \) be the identity maps. Since \( \pi_2(i_{2,\infty}^n) = 1 \) and \( \pi_2(i_{\infty,2}^n) = n^{1/2} \), we have
\[
n^{1/2} \leq w(\pi_2^d, \pi_2) (\text{id}_{l_\infty^n}) (\leq \gamma_2 (\text{id}_{l_\infty^n}) = n^{1/2}) .
\]
Now suppose that the quasi-Banach ideals of weak Hilbert operators and of weak weak Hilbert operators coincide. Hence there is a constant \( c \geq 0 \) such that
\[
n^{1/2} \leq w(\pi_2^d, \pi_2) (\text{id}_{l_\infty^n}) \leq cw(\pi_2^d, l_2^n, \infty) (\text{id}_{l_\infty^n}) \leq 8c\frac{n^{1/2}}{1 + \ln n} .
\]
But this is a contradiction and, therefore, the assertion is proved. □

Remark. We are able to prove the following result for diagonal operators \( D_\sigma : l_\infty \to l_\infty \), \( (x_k)_{k \in \mathbb{N}} \mapsto (\sigma_k x_k)_{k \in \mathbb{N}} \).

\( D_\sigma \) is a weak weak Hilbert operator if and only if \( \sup_{k \in \mathbb{N}} \frac{k^{1/2}}{1 + \ln k} \sigma_k^* < \infty \),

where \( (\sigma_k^*)_{k \in \mathbb{N}} \) denotes the nonincreasing rearrangement of \( \sigma \). On the other hand,

\( D_\sigma \) is a weak Hilbert operator if and only if \( \sup_{k \in \mathbb{N}} k^{1/2} \sigma_k^* < \infty \).

Remark. Note that Theorem 2 implies that on weak Hilbert spaces the dependence of the weak Hilbert norm \( w(\pi_2^d, \pi_2) \) and of the weak weak Hilbert norm \( w(\pi_2^d, l_2^n, \infty) \) cannot be linear.

References


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